Locality in integrable QFTs
characterization and explicit examples

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Partially joint work with H. Bostelmann

Topic

- What do I mean by "factorizing scattering models"?
  - Relativistic quantum field theory on 1+1 dimensional Minkowski space
  - A specific class of QFT: models with factorizing scattering matrix
  - Here: not in a thermodynamical (euclidean) setting, but interested in local observables of the QFT

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  - A specific class of QFT: models with factorizing scattering matrix
  - Here: not in a thermodynamical (euclidean) setting, but interested in local observables of the QFT
- Local observables
  - Physical meaning of "measurements"
  - Mathematically, linear bounded or unbounded operators associated with bounded regions in Minkowski space, so that operators associated with spacelike separated regions commute

Topic

- "Old" constructive approach: form factor program
  - Aimed at constructing pointlike quantum fields in terms of their matrix elements between asymptotic scattering states
  - Construct *n*-point functions as infinite series of integrals of these matrix elements
  - Problem: show the convergence of *n*-point functions

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- "Old" constructive approach: form factor program
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  - Construct n-point functions as infinite series of integrals of these matrix elements
  - Problem: show the convergence of n-point functions
- "New" constructive approach: wedge algebras (Schroer, . . . , Lechner)
  - Construct observables localized in wedges (easier to handle!)
  - Study properties of matrix elements between multi-particle states of non-free fields, described by deformed creators and annihilators satisfying a deformed version of the canonical commutation relations
  - Obtain local observables associated with bounded regions as the intersection of the respective sets of local observables associated with the "right" and "left" wedges
  - Proof that this intersection of algebras is non-trivial with a very abstract argument

Topic

- Here: How to obtain more information about the explicit form of the local observables
  - Study properties of certain matrix elements of these operators
  - Find a characterization of these operators in terms of these properties

Models with factorizing scattering matrix

Physical idea of the system:

- Imagine a system of spin-0 bosons of mass $\mu > 0$ moving in 1 spatial dimension.
- Two bosons (of different speed) will scatter – phase $S(\theta_1 - \theta_2)$ (θ "rapidity") is the two-particle scattering function.
- Multi-particle scattering is just a composition of subsequent 2-particle processes ("factorizing scattering matrix").
- $S = 1$: free field; $S = -1$: Ising model

Task: Given a function S, construct a corresponding quantum field theory.

Construction of QFTs with factorizing scattering matrix

The theory is constructed as a deformation of a free field.

- Zamolodchikov-Faddeev algebra:

$$\begin{aligned} z(\theta_1)z(\theta_2) &= S(\theta_1-\theta_2)\, z(\theta_2)z(\theta_1)\,, \\ z^\dagger(\theta_1)z^\dagger(\theta_2) &= S(\theta_1-\theta_2)\, z^\dagger(\theta_2)z^\dagger(\theta_1)\,, \\ z(\theta_1)z^\dagger(\theta_2) &= S(\theta_2-\theta_1)\, z^\dagger(\theta_2)z(\theta_1) + \delta(\theta_1-\theta_2)\cdot \mathbf{1}. \end{aligned}$$

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- Quantum fields: With $\hat{f}^{\pm}(\theta) = \int d^2x\, f(x)\, \exp(\pm ip(\theta)x)$, define

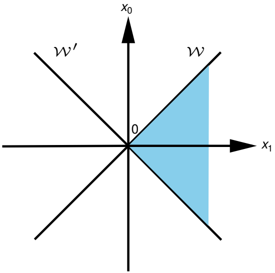
$$\phi(f) := z^\dagger(\hat{f}^+) + z(\hat{f}^-), \quad \phi'(f) := U(j)\phi(f^j)U(j).$$

Wedge-local fields

- The fields ϕ, ϕ' are wedge-local:
  - $\mathcal{W}$: standard right wedge; $\mathcal{W}'$: its causal complement (the left wedge). The fields fulfill

$$[\phi(f), \phi'(g)] = 0 \text{ if } \operatorname{supp} f \subset \mathcal{W}', \operatorname{supp} g \subset \mathcal{W}.$$

  - Interpretation: $\phi(x)$ an observable measurable in the infinite region $\mathcal{W}' + x$.



Local observables?

- We pass to the associated von Neumann algebras,

$$\mathfrak{A}(\mathcal{W}) = \{\exp i\phi(f) \mid \operatorname{supp} f \subset \mathcal{W}'\}'$$

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- For the standard double cone $\mathcal{O}_r = (\mathcal{W} - re_1) \cap (\mathcal{W}' + re_1)$, define

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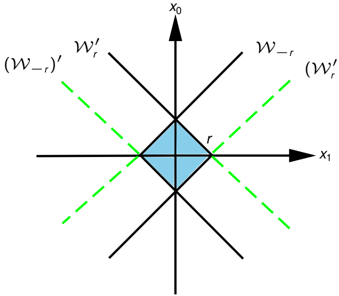
- This gives a consistent, covariant local net of algebras.
- Are there any observables localized in bounded regions (double cones)?
  - The fields $\phi(f)$, $\phi'(f)$ are not (except $S = 1$)
  - Polynomials of the fields are not (except certain polynomials in $S = -1$)
  - Must take limits of power series in order to obtain local operators.

Size of local algebras

- Is the intersection $\mathfrak{A}(\mathcal{O}_r)$ non-trivial?
  - Result (Lechner 2006): The vacuum is cyclic for $\mathfrak{A}(\mathcal{O}_r)$.
  - Uses a very abstract argument called "modular nuclearity condition" (analytic condition on the scattering function) and split property for wedge algebra inclusions.
  - This is enough to do scattering theory and compute the S matrix.
  - However, it does not give us explicit examples.



Araki's expansion

Araki's expansion for the free scalar Bose field ($S = 1$):
Every quadratic form A on Fock space (and therefore bounded and unbounded operators, as well) of a certain regularity class can be expanded as

$$A = \sum_{m,n=0}^{\infty} \int \frac{d^m\theta d^n\eta}{m!n!} f_{mn}(\boldsymbol{\theta}, \boldsymbol{\eta}) a^\dagger(\theta_1) \ldots a^\dagger(\theta_m) a(\eta_1) \ldots a(\eta_n)$$

Coefficient functions f_{mn} are given by

$$f_{mn}(\boldsymbol{\theta}, \boldsymbol{\eta}) = \big(\Omega, [a(\theta_1), \ldots, [a(\theta_m), [a^\dagger(\eta_1), \ldots, [a^\dagger(\eta_n), A] \ldots]\Omega\big).$$

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What if A is localized in the standard double cone $\mathcal{O}_r$? [$A \in \mathcal{A}(\mathcal{O}_r)$]

- Write a, $a^\dagger$ as Fourier transforms of time-zero fields ϕ, π.
- Basically, f_{mn} become Fourier transforms of functions of compact support.

Characterization of local operators in the free field

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If A is local in $\mathcal{O}_r$, then there exist entire functions $F_k : \mathbb{C}^k \to \mathbb{C}$ such that

$$f_{mn}(\boldsymbol{\theta}, \boldsymbol{\eta}) = F_{m+n}(\theta_1, \ldots, \theta_m, \eta_1 + i\pi, \ldots, \eta_n + i\pi).$$

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The F_k have the properties:

- They are symmetric in their arguments (due to the CCR),
- They are $2i\pi$-periodic in each argument (due to $p(\theta)$ being periodic),
- They fulfill certain, r-dependent bounds in imaginary directions (Paley-Wiener).

These conditions are "if and only if" on the level of quadratic forms (remark on operator domains later).

Generalization of Araki's expansion

In our interacting models ($S \neq 1$), we can expand every quadratic form of a certain regularity class as

$$A = \sum_{m,n=0}^{\infty} \int \frac{d^m\theta d^n\eta}{m!n!} f_{mn}(\boldsymbol{\theta}, \boldsymbol{\eta}) z^{\dagger m}(\boldsymbol{\theta}) z^n(\boldsymbol{\eta}).$$

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- What are the coefficient functions f_{mn}?
  - General formula in terms of contracted matrix elements of A

$$f_{m,n}^{[A]} := \sum_{C \in \mathcal{C}_{m,n}} (-1)^{|C|} \delta_C S_C(\theta, \eta) \langle \ell_C(\theta), A r_C(\eta) \rangle$$

  - For special S-matrices (Lechner and Grosse 2007) a commutator formula from above works with $a, a^\dagger$ replaced by $z, z^\dagger$ and with $[\cdot, \cdot]$ replaced by a certain "deformed commutator" $[\cdot, \cdot]_S$.

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- We can compute the effect of
  - Poincaré transformations,
  - space-time reflections

  on the coefficients f_{mn}.

Main result: Characterization of local operators in the interacting case

If the quadratic form A is local in $\mathcal{O}_r$, then there exist meromorphic functions $F_k : \mathbb{C}^k \to \mathbb{C}$ such that

$$f_{mn}(\boldsymbol{\theta}, \boldsymbol{\eta}) = F_{m+n}(\theta_1 + i0, \ldots, \theta_m + i0, \eta_1 + i\pi - i0, \ldots, \eta_n + i\pi - i0).$$

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Again, these conditions are "if and only if" on the level of quadratic forms (remark on operator domains later).

How to obtain the equivalence

Very brief sketch!

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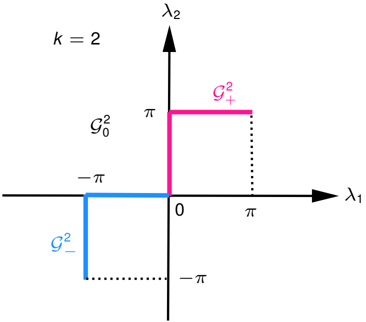
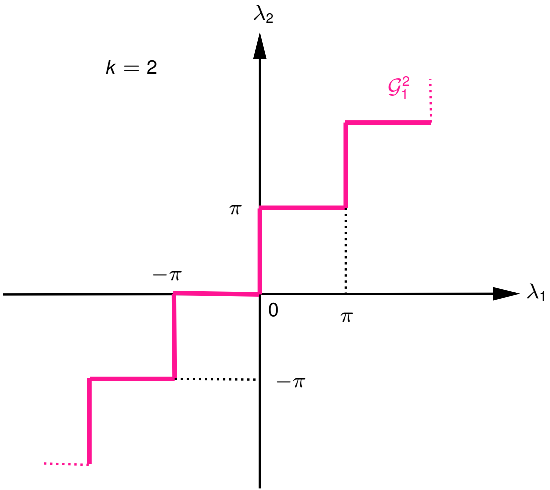
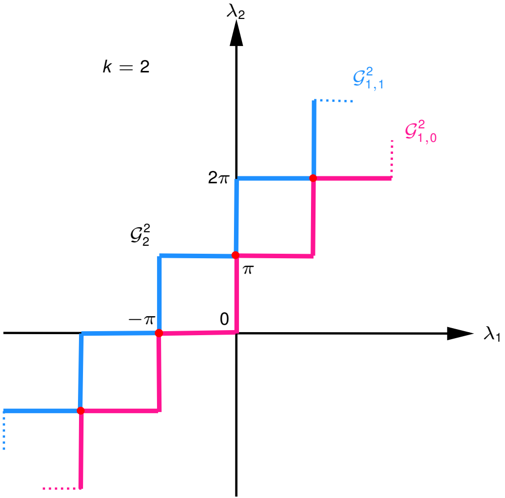
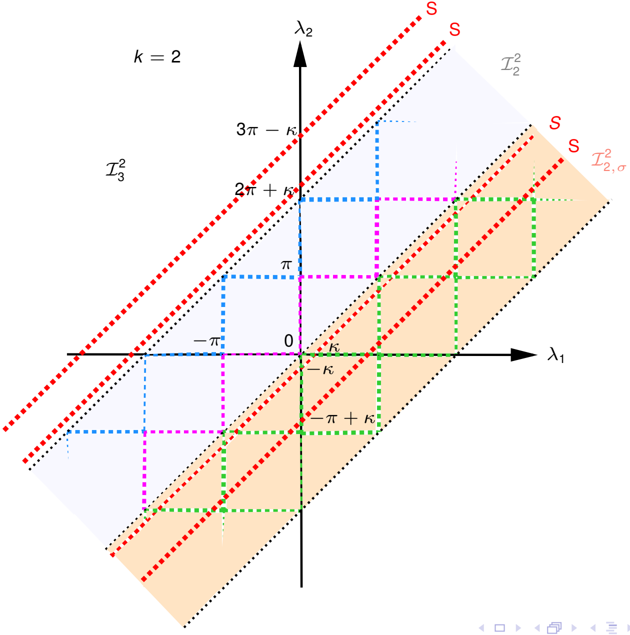
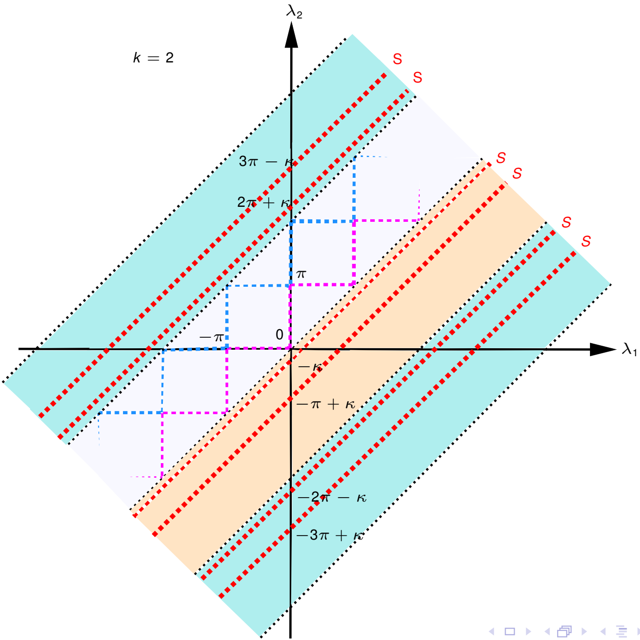
- From local A to meromorphic F_k:
  - We know from Lechner (2008) that the $f_{mn}[A]$ have an analytic continuation to a certain domain, due to wedge-locality of A.
  - $U(j)A^*U(j)$ is wedge-local as well (A local in the opposite wedge), so $f_{mn}[U(j)A^*U(j)]$ extends to another analytic function.
  - Stitch these together to obtain F_k.




$k = 2$
λ_2
λ_1
$\mathcal{G}_1^2$
π
$-\pi$
0
π
$-\pi$


λ_2
$k = 2$
$\mathcal{G}^2_{1,1}$
$\mathcal{G}^2_{1,0}$
2π
$\mathcal{G}^2_2$
π
$-\pi$
0
λ_1


k = 2
λ_2
λ_1
S
S
S
S
$\mathcal{I}_2^2$
$\mathcal{I}_3^2$
$\mathcal{I}_{2,\sigma}^2$
$3\pi - \kappa$
$2\pi + \kappa$
π
$-\pi$
0
κ
$-\kappa$
$-\pi + \kappa$



λ_2
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S
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S
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$-\kappa$
$-\pi + \kappa$
$-2\pi - \kappa$
$-3\pi + \kappa$

How to obtain the equivalence

- From meromorphic F_k to local A:
  - Show that $A = \sum_{mn} \int F_{m+n}(\ldots) z^{\dagger m} z^n$ is local in a shifted left wedge (compute commutator with $\phi'(f)$; shift integral contours; use bounds on F_k)
  - Show that $U(j) A^* U(j)$ is given by $F_k(\cdot + i\pi)$; this involves periodicity and value of residues.
  - $F_k(\cdot + i\pi)$ fulfills the same bounds, thus A is local in a shifted right wedge.

Quadratic forms and locality

- Question is "convergence of the series" – note that it's infinite in general, since Res $F_k \sim F_{k-2}$.
- Smeared annihilators/creators are unbounded; impossible to read off $\|A\|$ from estimates for F_k.

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   - The expansion series becomes a finite sum in matrix elements.
2. We have a full converse of the characterization.
3. Allows to include local fields from the form factor programme, $\phi_{\mathrm{local}}(f)$, which are unbounded.

Quadratic forms and locality

We consider quadratic forms of a certian "regularity class" ($\mathcal{Q}^{\omega}$), where the singular behaviour of A is somehow "controlled":

$$\|A\|_k^{\omega} = \frac{1}{2}\|Q_k A e^{-\omega(H/\mu)} Q_k\| + \frac{1}{2}\|Q_k e^{-\omega(H/\mu)} A Q_k\| < \infty$$

ω is a function of energy of suitable growth.

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Definition

$A \in \mathcal{Q}^\omega$ is called ω-local in the standard wedge $\mathcal{W}$ iff

$$(\phi(f)^* \psi, A\chi) = (\psi, A\phi(f)\chi) \quad \text{whenever } f \in \mathcal{D}^\omega(\mathcal{W}'); \ \psi, \chi \in \mathcal{H}^{\omega,\mathrm{f}}.$$

ω-locality in other wedges and double cones is defined correspondingly.

Quadratic forms and locality

We consider quadratic forms of a certian "regularity class" ($\mathcal{Q}^\omega$), where the singular behaviour of A is somehow "controlled":

$$\|A\|_k^\omega = \frac{1}{2}\|Q_k A e^{-\omega(H/\mu)} Q_k\| + \frac{1}{2}\|Q_k e^{-\omega(H/\mu)} A Q_k\| < \infty$$

ω is a function of energy of suitable growth.
We need a notion of locality adapted to quadratic forms.

Definition

$A \in \mathcal{Q}^\omega$ is called ω-local in the standard wedge $\mathcal{W}$ iff

$$(\phi(f)^*\psi, A\chi) = (\psi, A\phi(f)\chi) \quad \text{whenever } f \in \mathcal{D}^\omega(\mathcal{W}');\ \psi, \chi \in \mathcal{H}^{\omega,\mathrm{f}}.$$

ω-locality in other wedges and double cones is defined correspondingly.

So, our characterization actually works on the level of ω-local quadratic forms.

Weak locality vs. locality / Quadratic forms vs. operators

- The characterization theorem works best for ω-local quadratic forms.
- For QFT, we want operators which are local in the usual sense.
- Relation between these notions?

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Lemma

Let A be a closed operator with core $\mathcal{H}^{\omega,\mathrm{f}}$ such that $\mathcal{H}^{\omega,\mathrm{f}} \subset \operatorname{dom} A^*$. Suppose that

$$\forall g \in \mathcal{D}^{\omega}_{\mathbb{R}}(\mathbb{R}^2) : \; \exp(i\phi(g)^-)\mathcal{H}^{\omega,\mathrm{f}} \subset \operatorname{dom} A. \tag{1}$$

Then A is ω-local in $\mathcal{O}_r$ iff it is affiliated with $\mathcal{A}(\mathcal{O}_r)$.

Weak locality vs. locality / Quadratic forms vs. operators

Lemma

Let $A \in \mathcal{Q}^\omega$. Suppose that

$$\sum_{m,n=0}^{\infty} \frac{2^{(m+n)/2}}{\sqrt{m!n!}} \|f_{m,n}^{[A]}\|_{m\times n}^{\omega} < \infty.$$

Then, A extends to a closed operator A^- with core $\mathcal{H}^{\omega,\mathrm{f}}$; one has $\mathcal{H}^{\omega,\mathrm{f}} \subset \mathrm{dom}(A^-)^*$. Also, the condition (1) is fulfilled by A^-.

Examples

- Buchholz-Summers type:
  Let $F_k = 0$ for $k \neq 2$, we set
  $$F_2(\zeta_1, \zeta_2) = \sinh\left(\frac{\zeta_1 - \zeta_2}{2}\right) \tilde{g}(\mu E(\zeta)),$$
  where $\tilde{g}$ is the Fourier transform of a function $g \in \mathcal{D}(-r, r)$ for some $r > 0$, with $\omega(p) := \ell \log(1 + p)$, ℓ sufficiently large.
- Schroer-Truong type:
  Let $g \in \mathcal{D}(\mathbb{R})$, $g \in \mathcal{S}_\omega$ with $\omega(p) = p^\alpha$, $1/3 < \alpha < 1$. We set ($k \in \mathbb{N}_0$)
  $$F_{2k+1}(\zeta) = \frac{1}{(4\pi i)^k k!} \tilde{g}(\mu E(\zeta)) \sum_{\sigma \in \mathfrak{S}_{2k+1}} \operatorname{sign} \sigma \prod_{j=1}^{k} \tanh \frac{\zeta_{\sigma(2j-1)} - \zeta_{\sigma(2j)}}{2},$$
  with $F_{2k} = 0$ for any k.

Results & Open Questions

Results:

- In factorizing scattering models, there is an analogue to the Araki expansion.
- ω-locality of quadratic forms can be characterized by analyticity and boundedness properties of the expansion coefficients.
- We can extend the quadratic forms to closed, possibly unbounded, operators affiliated with the local algebras of bounded operators.

Results & Open Questions

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- In factorizing scattering models, there is an analogue to the Araki expansion.
- ω-locality of quadratic forms can be characterized by analyticity and boundedness properties of the expansion coefficients.
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Open questions:

- Investigate examples for general S.
- Show expansion of all local operators into (interacting) pointlike objects.
- Generalize our analysis to arbitrary spacetime dimensions, to a richer particle spectrum, to theories where the scattering function can have poles on the physical strip,...