Quantum field theory on affine bundles

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Why affine bundles?

Interesting for inhomogeneous field equations, whose space of solutions is naturally an affine space.
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Yang-Mills theory:

- *Principal bundle* over a globally hyperbolic spacetime;
- Fields are represented by **connections**, which **are sections of an affine bundle**;
- *Gauge group* as symmetry group (the hardest part).

Simplest case: **The Maxwell field**.

Affine stuff: Spaces

An **affine space** (A, V, Φ) modeled over the vector space V is a set A endowed with a free and transitive right group action $\Phi : A \times V \to A$ of the abelian group $(V, +)$.

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Euristically:
A vector space where we forgot which one is the null vector.

In fact, fixing an element of A, we can endow A with a vector structure. Now A, as a vector space, becomes isomorphic to V.

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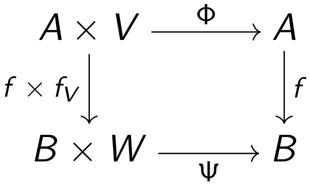
$(A, V, \Phi), (B, W, \Psi)$ affine spaces. A map $f : A \to B$ is an **affine morphism** if there exists a linear map $f_V : V \to W$ such that $f(\Phi(a, v)) = \Psi(f(a), f_V(v))$.



Affine stuff: Spaces

An **affine space** (A, V, Φ) modeled over the vector space V is a set A endowed with a free and transitive right group action $\Phi : A \times V \to A$ of the abelian group $(V, +)$.

Trivial example:
A vector space V may be regarded as an affine space $(V, V, +)$ modeled on itself.

Affine stuff: Spaces

Vector dual $A^\dagger$ of an affine space (A, V, Φ):
The set of all affine morphisms from (A, V, Φ) to the vector space $\mathbb{R}$ regarded as an affine space modeled on itself.

Because of the vector structure of the target space $\mathbb{R}$*, this set comes naturally endowed with a vector space structure.*

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The **dual** $f^\dagger : A^\dagger \to B^\dagger$ of an affine isomorphism $f : A \to B$ is defined by $f^\dagger(a^\dagger) = a^\dagger \circ f^{-1}$ for each $a^\dagger \in A^\dagger$.

$f^\dagger$ automatically turns out to be a linear map.

Affine stuff: Bundles

Affine bundle $(\mathsf{A}, \mathsf{V}, M)$:

- Fiber bundle $(\mathsf{A}, M, \pi_{\mathsf{A}})$ with an affine space (A, V, Φ) as fiber;
- Vector bundle $(\mathsf{V}, M, \pi_{\mathsf{V}})$ with V as typical fiber;
- Trivializations of A have the **affine property** wrt those of V.

Affine property:
$\forall x \in M$ there exists a neighborhood U of x, a trivialization $\mathsf{A}|_U \stackrel{\phi}{\to} U \times A$ of A and a trivialization $\mathsf{V}|_U \stackrel{\phi_V}{\to} U \times V$ of V such that, for each $y \in M$, $\mathsf{A}|_y \stackrel{\phi\restriction_y}{\to} A$ is an affine isomorphism whose linear part is the vector space isomorphism $\mathsf{V}|_y \stackrel{\phi_V\restriction_y}{\to} V$.

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Vector dual $(\mathsf{A}^{\dagger}, M, \pi_{\mathsf{A}^{\dagger}})$ *of an affine bundle:*
Consider the Hom-bundle from the affine bundle $(\mathsf{A}, \mathsf{V}, M)$ to the vector bundle $M \times \mathbb{R}$ regarded as an affine bundle.

We are simply taking the vector dual fiberwise.

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$(\mathsf{A}, \mathsf{V}, M), (\mathsf{B}, \mathsf{W}, N)$ affine bundles. A bundle morphism $(f, \underline{f}) : (\mathsf{A}, M, \pi_{\mathsf{A}}) \to (\mathsf{B}, N, \pi_{\mathsf{B}})$ is an **affine bundle morphism** if $\mathsf{A}|_x \overset{f\restriction_x}{\to} \mathsf{B}|_{\underline{f}(x)}$ is an affine isomorphism $\forall x \in M$.
Induced vector bundle morphism: $(\mathsf{V}, M, \pi_{\mathsf{V}}) \overset{(f_V, \underline{f})}{\to} (\mathsf{W}, N, \pi_{\mathsf{W}})$.

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Remark:
The space $\Gamma(M, \mathsf{A})$ of sections of the fiber bundle $(\mathsf{A}, M, \pi_{\mathsf{A}})$ (which is never empty) is an affine space modeled on the vector space $\Gamma(M, \mathsf{V})$ of sections of the vector bundle $(\mathsf{V}, M, \pi_{\mathsf{V}})$.

Affine stuff: Differential operators

$(\mathsf{A}, \mathsf{V}, M)$ affine bundle, $(\mathsf{W}, M, \pi_{\mathsf{W}})$ vector bundle. An **affine differential operator** $P : \Gamma(M, \mathsf{A}) \to \Gamma(M, \mathsf{W})$ is an affine morphism whose linear part $P_V : \Gamma(M, \mathsf{V}) \to \Gamma(M, \mathsf{W})$ is a differential operator in the usual sense.

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P is *formally adjoinable* if there exists a differential operator $P^* : \Gamma(M, \mathsf{W}^*) \to \Gamma(M, \mathsf{A}^\dagger)$ such that for each $w^* \in \Gamma_c(M, \mathsf{W}^*)$ and for each $\sigma \in \Gamma(M, \mathsf{A})$ the following holds:

$$\int_M \mathrm{vol}_M \,(P^* w^*)(\sigma) = \int_M \mathrm{vol}_M \, w^*(P\sigma).$$

Affine stuff: Differential operators

Theorem: Each affine diff. op. $P : \Gamma(M, \mathrm{A}) \to \Gamma(M, \mathrm{W})$ is formally adjoinable, but its formal adjoint is *not unique*.
If P^* and $P^{*\prime}$ are both formal adjoints of P, there exists a differential operator $Q : \Gamma(M, \mathrm{W}^*) \to \mathrm{C}^\infty(M)$ such that $P^{*\prime} - P^* = Q\mathbb{1}$ and $\int_M \mathrm{vol}_M \, Q w^* = 0 \;\; \forall w^* \in \Gamma_c(M, \mathrm{W}^*)$.

Remarks:

- $\mathbb{1} \in \Gamma(M, \mathrm{A}^\dagger)$ is defined by $\mathbb{1}(a) = 1$ for each $a \in \mathrm{A}$.
- This non-uniqueness can be eliminated modding out an appropriate vector space *(see later)*.

Classical dynamics

From now on:

- Only globally hyperbolic spacetimes as base manifolds;
- Maps between bases are causal embeddings;
- Vector bundles are endowed with an inner product;
- Vector bundle morphisms preserve the inner products.

In particular the first and third statements apply to the base manifold and the vector bundle underlying a given affine bundle, while the second and the fourth apply to the base map and the linear part of an affine bundle morphism.

Classical dynamics: Green-hyperbolic operators

$(\mathsf{A}, \mathsf{V}, M)$ affine bundle. An affine differential operator $P : \Gamma(M, \mathsf{A}) \to \Gamma(M, \mathsf{V})$ is **affine Green-hyperbolic** if its linear part $P_V : \Gamma(M, \mathsf{V}) \to \Gamma(M, \mathsf{V})$ is a Green-hyperbolic differential operator in the usual sense.

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Remark: There exist many formal adjoints of P!
Let $P^* : \Gamma(M, \mathrm{V}^*) \to \Gamma(M, \mathrm{A}^\dagger)$ be one of the adjoints. Then

$$\mathrm{Adj}(P) = P^* + \mathbb{1} \left\{ \begin{array}{l} Q : \Gamma(M, \mathrm{V}^*) \to \mathrm{C}^\infty(M) \text{ such that} \\ \int_M \mathrm{vol}_M \, Q v^* = 0 \;\; \forall v^* \in \Gamma_c(M, \mathrm{V}^*) \end{array} \right\} .$$

is the set of the formal adjoints of P.

Classical dynamics: Observables

Space of **observables** $\mathrm{Obs}(\mathsf{A}, \mathsf{V}, M) = \{F_\phi : \phi \in \Gamma_c(M, \mathsf{A}^\dagger)\}$ defined via the map F introduced below.

$$F : \phi \in \Gamma_c(M, \mathsf{A}^\dagger) \mapsto F_\phi = \int_M \mathrm{vol}_M\, \phi(\cdot) : \Gamma(M, \mathsf{A}) \to \mathbb{R}.$$

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Theorem: $\Gamma_c(M, \mathsf{A}^\dagger)$ is separating on $\Gamma(M, \mathsf{A})$, but the converse does not hold. More precisely:

- If $F_\phi(\sigma) = F_\phi(\sigma')$ for each $\phi \in \Gamma_c(M, \mathsf{A}^\dagger)$ then $\sigma = \sigma'$;
- If $F_\phi(\sigma) = 0$ for each $\sigma \in \Gamma(M, \mathsf{A})$ then $\phi = a\mathbb{1}$ with $a \in \mathrm{C}_c^\infty(M)$ such that $\int_M \mathrm{vol}_M \, a = 0$.

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Remark:

According to the theorem, **trivial observables** are generated by

$$\mathrm{Triv}(\mathsf{A}, \mathsf{V}, M) = \left\{ a \in \mathrm{C}^\infty_c(M) : \int_M \mathrm{vol}_M\, a = 0 \right\} \mathbb{1} \subseteq \Gamma_c(M, \mathsf{A}^\dagger).$$

Classical dynamics: $\mathrm{Adj}(P)$ and $\mathrm{Triv}(\mathsf{A}, \mathsf{V}, M)$

- Set of formal adjoints of $P : \Gamma(M, \mathsf{A}) \to \Gamma(M, \mathsf{V})$:

$$\mathrm{Adj}(P) = P^* + \mathbb{1} \left\{ \begin{array}{l} Q : \Gamma(M, \mathsf{V}^*) \to \mathrm{C}^\infty(M) \text{ such that} \\ \int_M \mathrm{vol}_M \, Q v^* = 0 \;\; \forall v^* \in \Gamma_c(M, \mathsf{V}^*) \end{array} \right\};$$

- Set generating trivial observables on $(\mathsf{A}, \mathsf{V}, M)$:

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Modding out $\mathrm{Triv}(\mathsf{A}, \mathsf{V}, M)$ we obtain a **unique** formal adjoint $P^* : \Gamma_c(M, \mathsf{V}^*) \to \Gamma_c(M, \mathsf{A}^\dagger)/\mathrm{Triv}(\mathsf{A}, \mathsf{V}, M)$!

Classical dynamics: Phase space

- The linear part P_V of P is **formally self-adjoint**;
- Identify $(\mathsf{V}, M, \pi_{\mathsf{V}})$ with its dual using the inner product;
- Consider **Green operators** $G^{\pm}$ for P_V and introduce the corresponding causal propagator $G = G^{+} - G^{-}$.

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Bilinear form on $\Gamma_c(M, \mathsf{A}^{\dagger})/\mathrm{Triv}(\mathsf{A}, \mathsf{V}, M)$:

$$(\phi, \psi) \in \left(\frac{\Gamma_c(M, \mathsf{A}^{\dagger})}{\mathrm{Triv}(\mathsf{A}, \mathsf{V}, M)}\right)^2 \mapsto \int_M \mathrm{vol}_M \, \langle \phi_V, G\psi_V \rangle_{\mathsf{V}}.$$

Well defined since $\mathrm{Triv}(\mathsf{A}, \mathsf{V}, M)$ does not affect linear parts.

Classical dynamics: Phase space

Remarks:

- $(P^*\phi)_V = P_V\phi_V$ for each $\phi \in \Gamma_c(M, \mathsf{A}^\dagger)/\mathrm{Triv}(\mathsf{A}, \mathsf{V}, M)$;
- $P_V(\Gamma_c(M, \mathsf{V})) = \ker(G)$, hence we can take the quotient over the subset $P^*(\Gamma_c(M, \mathsf{V})) \subset \Gamma_c(M, \mathsf{A}^\dagger)/\mathrm{Triv}(\mathsf{A}, \mathsf{V}, M)$:

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Pairing between observables:

$$\tau : \mathcal{E} \times \mathcal{E} \to \mathbb{R}, \quad ([\phi], [\psi]) \mapsto \int_M \mathrm{vol}_M \langle \phi_V, G\psi_V \rangle_{\mathsf{V}},$$

where $\mathcal{E} = (\Gamma_c(M, \mathsf{A}^\dagger)/\mathrm{Triv}(\mathsf{A}, \mathsf{V}, M))/P^*(\Gamma_c(M, \mathsf{V}))$.

Categorical formulation: Aff

Object $(\mathsf{A}, \mathsf{V}, M, P)$**:**

- Affine bundle $(\mathsf{A}, \mathsf{V}, M)$;
- Vector bundle $(\mathsf{V}, M, \pi_{\mathsf{V}})$ endowed with a non-degenerate bilinear form $\langle \cdot, \cdot \rangle_{\mathsf{V}}$;
- Globally hyperbolic spacetime M;
- Affine Green-hyperbolic differential operator $P : \Gamma(M, \mathsf{A}) \to \Gamma(M, \mathsf{V})$ with formally self-adjoint linear part $P_V : \Gamma(M, \mathsf{V}) \to \Gamma(M, \mathsf{V})$.

Categorical formulation: Aff

Morphism $(f, \underline{f})$**:**

- Affine bundle morphism $(f, \underline{f}) : (\mathsf{A}_1, \mathsf{V}_1, M_1) \to (\mathsf{A}_2, \mathsf{V}_2, M_2)$;
- The linear part $(f_V, \underline{f}) : (\mathsf{V}_1, M_1, \pi_{\mathsf{V}_1}) \to (\mathsf{V}_2, M_2, \pi_{\mathsf{V}_2})$ preserves the inner products;
- $\underline{f} : M_1 \to M_2$ is a causal embedding;
- The following diagram commutes:

$$\begin{array}{ccc} \Gamma(M_2, \mathsf{A}_2) & \xrightarrow{P_2} & \Gamma(M_2, \mathsf{V}_2) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle {f_V}^*} \\ \Gamma(M_1, \mathsf{A}_1) & \xrightarrow[P_1]{} & \Gamma(M_1, \mathsf{V}_1) \end{array}$$

Categorical formulation: Vec

Object $(V, \langle\cdot,\cdot\rangle_V)$**:**

- Vector space V endowed with a bilinear form $\langle\cdot,\cdot\rangle_V$.

Categorical formulation: Vec

Object $(V, \langle \cdot, \cdot \rangle_V)$:

- Vector space V endowed with a bilinear form $\langle \cdot, \cdot \rangle_V$.

Morphism $L : (V_1, \langle \cdot, \cdot \rangle_{V_1}) \to (V_2, \langle \cdot, \cdot \rangle_{V_2})$:

- Injective linear map $L : V_1 \to V_2$ preserving the bilinear forms:

$$
\begin{array}{ccc}
V_1 \times V_1 & \xrightarrow{L \times L} & V_2 \times V_2 \\
& \searrow^{\langle \cdot, \cdot \rangle_{V_1}} & \downarrow \langle \cdot, \cdot \rangle_{V_2} \\
& & \mathbb{R}
\end{array}
$$

Categorical formulation: The functor $\mathfrak{PhSp}$

Theorem:

- For each object $(\mathsf{A}, \mathsf{V}, M, P)$ in Aff, the associated phase space $(\mathcal{E}, \tau)$ constructed above is an **object** in Vec;
- For each $(f, \underline{f}) : (\mathsf{A}_1, \mathsf{V}_1, M_1, P_1) \to (\mathsf{A}_2, \mathsf{V}_2, M_2, P_2)$ in Aff, the map
$$\mathcal{E}_1 \to \mathcal{E}_2, \quad [\phi] \mapsto [(f^\dagger)_* \phi],$$
where $(f^\dagger, \underline{f}) : ({\mathsf{A}_1}^\dagger, M_1, \pi_{{\mathsf{A}_1}^\dagger}) \to ({\mathsf{A}_2}^\dagger, M_2, \pi_{{\mathsf{A}_2}^\dagger})$ is defined by $f^\dagger \restriction_x (a^\dagger) = a^\dagger \circ (f \restriction_x)^{-1} \;\; \forall x \in M_1, \forall a^\dagger \in {A_1}^\dagger|_x$ and ${f^\dagger}_*$ is the usual pushforward on compactly supported sections, is a **morphism** in Vec (injective and bilinear-forms-preserving).

Categorical formulation: The functor $\mathfrak{PhSp}$

- Send $(\mathsf{A}, \mathsf{V}, M, P)$ in Aff to $(\mathcal{E}, \tau)$ in Vec;
- Send $(f, \underline{f}) : (\mathsf{A}_1, \mathsf{V}_1, M_1, P_1) \to (\mathsf{A}_2, \mathsf{V}_2, M_2, P_2)$ in Aff to $(f^\dagger)_* : (\mathcal{E}_1, \tau_1) \to (\mathcal{E}_2, \tau_2)$ in Vec.

Theorem:
The assignment above is functiorial. Specifically, it defines a **covariant functor** $\mathfrak{PhSp} : \mathsf{Aff} \to \mathsf{Vec}$ which fulfils:
- **Causality property**;
- **Time-slice axiom**.

Quantization: Bosons

- Subcategory Aff^B encompassing all objects in Aff with a **symmetric inner product**;
- Subcategory Vec^B encompassing all objects in Vec with a **skew-symmetric bilinear form**;

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Theorem:
$\mathfrak{PhSp}$ restricts to a covariant functor $\mathfrak{PhSp}^B : \mathsf{Aff}^B \to \mathsf{Vec}^B$ fullfilling the causality property and the time-slice axiom.
Moreover composing with the usual bosonic quantization functor $\mathfrak{CCR} : \mathsf{Vec}^B \to *\mathsf{Alg}$ gives rise to a **bosonic locally covariant quantum fied theory**.

Quantization: Fermions

- Subcategory Aff^F encompassing all objects in Aff with a **skew-symmetric inner product**;
- Subcategory Vec^F encompassing all objects in Vec with a **symmetric bilinear form**.

Theorem:
$\mathfrak{PhSp}$ restricts to a covariant functor $\mathfrak{PhSp}^F : \mathsf{Aff}^F \to \mathsf{Vec}^F$ fullfilling the causality property and the time-slice axiom. Moreover composing with the usual fermionic quantization functor $\mathfrak{CAR} : \mathsf{Vec}^F \to *\mathsf{Alg}$ gives rise to a **fermionic locally covariant quantum field theory**.

Induction of states

Remark:
One can consistently consider the linear part of affine field theories obtaining locally covariant quantum field theories in the usual sense.

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Strategy:
Find simple algebra morphisms from $\mathcal{A}^B(\mathrm{A}, \mathrm{V}, M, P)$ to $\mathcal{A}^B_{lin}(\mathrm{A}, \mathrm{V}, M, P)$ to induce states on the full affine algebra from states on the linearized algebra via pull-back.

Induction of states: Morphisms in $*\mathsf{Alg}$

For each section $s \in \Gamma(M, \mathsf{A})$ such that $P(s) = 0$, one can define a particular morphism κ_s in $*\mathsf{Alg}$, which keeps somewhat track of the affine part.

The definition is given on generators of the algebras involved:

$$\begin{aligned} \kappa_s : \mathcal{A}^B(\mathsf{A}, \mathsf{V}, M, P) &\to \mathcal{A}^B_{lin}(\mathsf{A}, \mathsf{V}, M, P), \\ \Psi([\phi]) &\mapsto \Psi_{lin}([\phi_V]) + \int_M \mathrm{vol}_M \, \phi(s) \, \mathbb{1}. \end{aligned}$$

Induction of states: Pull-back

For each $s \in \Gamma(M, \mathsf{A})$ such that $P(s) = 0$ and each state ω on $\mathcal{A}^B_{lin}(\mathsf{A}, \mathsf{V}, M, P)$, we can define a **state** $\omega_{\kappa_s} = \omega \circ \kappa_s$ **on** $\mathcal{A}^B(\mathsf{A}, \mathsf{V}, M, P)$ **by pull-back via** κ_s.

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Property:
Even when ω is quasi-free, ω_{κ_s} is not, since

$$\omega_{\kappa_s}(\Psi([\phi])) = \int_M \mathrm{vol}_M \, \phi(s) \neq 0.$$

This allows us to **measure the source** when dealing with inhomogeneous field equations *(see later)*.

Induction of states: μSC

We say that a state ω on $\mathcal{A}^B(\mathsf{A}, \mathsf{V}, M, P)$ fulfils the microlocal spectrum condition (μSC) when the wave-front set $\mathrm{WF}(\omega_n)$ of any of its n-point functions is included in Γ_n.

$$\Gamma_n = \left\{ \begin{array}{r} (x_1, \zeta_1; \ldots; x_n, \zeta_n) \in \mathsf{T}^* M^n \setminus \mathcal{Z} : \text{ there exists a graph} \\ G \in \mathsf{G}_n \text{ and an immersion of } G \text{ into } M \text{ such} \\ \text{that } \zeta_i = \sum_{\gamma_r(i,j)}^{i<j} k_r(x_i) - \sum_{\gamma_r(i,j)}^{i>j} k_r(x_i). \end{array} \right\}$$

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Theorem: Take ω on $\mathcal{A}^B_{lin}(\mathsf{A}, \mathsf{V}, M, P)$ quasi-free Hadamard and $s \in \Gamma(M, \mathsf{A})$ such that $P(s) = 0$. Then the state ω_{κ_s} on $\mathcal{A}^B(\mathsf{A}, \mathsf{V}, M, P)$ fulfils the microlocal spectrum condition.

Example: Inhomogeneous matter field theory

- Vector bundle $(\mathsf{V}, M, \pi_{\mathsf{V}})$ over a globally hyperbolic spacetime M regarded as an affine bundle $(\mathsf{V}, \mathsf{V}, M)$ modeled on itself;
- Non-degenerate bilinear form on $(\mathsf{V}, M, \pi_{\mathsf{V}})$;
- Formally self-adjoint Green-hyperbolic differential operator P_V acting on $\Gamma(M, \mathsf{V})$;
- Section $J \in \Gamma(M, \mathsf{V})$.

$P = P_V - J\mathbb{1}$: is an affine Green-hyperbolic operator on $\Gamma(M, \mathsf{V})$ whose linear part P_V is formally self-adjoint.

We can apply the affine machinery!

Example: Observables

Type 1 For each $s \in \Gamma(M, \mathrm{V})$ such that $P(s) = 0$ and each $h \in \Gamma_c(M, \mathrm{V})$, take $\phi = \langle h, \cdot - s \rangle_V \in \Gamma_c(M, \mathrm{V}^\dagger)$.

F_ϕ measures flactuations around the solution s.

No information about the source J!

Example: Observables

Type 1 For each $s \in \Gamma(M, \mathrm{V})$ such that $P(s) = 0$ and each $h \in \Gamma_c(M, \mathrm{V})$, take $\phi = \langle h, \cdot - s \rangle_V \in \Gamma_c(M, \mathrm{V}^\dagger)$.

F_ϕ measures flactuations around the solution s.

No information about the source J!

Type 2 For $h \in \Gamma_c(M, \mathrm{V})$, take $\psi = \langle P_V h, \cdot \rangle_V \in \Gamma_c(M, \mathrm{V}^\dagger)$.

If $s \in \Gamma(M, \mathrm{V})$ is a solution, $P_V s = J$, and hence

$$F_\psi(s) = \int_M \mathrm{vol}_M \langle h, J \rangle_V .$$

Affine theories can measure sources!

Conclusions and perspectives

- **Locally covariant QFTs on affine bundles**;
- **Well-behaved states from usual states**;
- Relevant cases:
  Inhomogenous field equations,
  Maxwell field can be treated in this context
  [MB, C. Dappiaggi, A. Schenkel, *work in progress*];
- Main advantage of affine theories:
  Observables which allow the complete reconstruction of the source exist!

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Thank you for you attention!