(Work in progress on) Quantum Field Theory on Curved Supermanifolds

Paul-Thomas Hack

Universität Hamburg

LQP 31, Leipzig, November 24, 2012

in collaboration with Andreas Schenkel

Motivation

- supergravity theories can be described as field theories on supermanifolds = "curved superspace", invariant under "superdiffeomorphisms"
- want to obtain a systematic treatment of supergravity theories as "locally superinvariant" algebraic QFT on curved supermanifolds (CSM)
- long term goal: understand SUSY non-renormalisation theorems in the algebraic language and extend them to curved spacetimes
- first: understand Wess-Zumino model as QFT on CSM covariant under "superdiffeomorphisms" $\rightarrow$ locally supercovariant QFT

Supersymmetry and superspace for pedestrians

- a simple SUSY model (Wess-Zumino model) in $d = 2+1$ Minkowski spacetime $\mathbb{R}^3$: ϕ, ψ, F

invariant under : $\phi \mapsto \phi + \bar{\varepsilon}\psi \qquad \psi \mapsto \psi + \partial\!\!\!/\phi\varepsilon + F\varepsilon \qquad F \mapsto F + \bar{\varepsilon}\partial\!\!\!/\psi$

on shell : $\Box\phi = 0 \qquad \partial\!\!\!/\psi = 0 \qquad F = 0$

- idea: consider $\Phi = (\phi, \psi, F)$ as a single "superfield" on "superspace" $\mathbb{R}^{3|2} \ni (x^\alpha, \theta_a)$

$$\Phi(x,\theta) = \phi(x) + \overline{\psi}(x)\theta + \frac{1}{2}F(x)\theta^2 \qquad (\theta^2 = \overline{\theta}\theta)$$

$$\Phi'(x,\theta) = \Phi(x',\theta')$$

"special supertranslations" $\quad x^\alpha \mapsto x'^\alpha = x^\alpha + \bar{\varepsilon}\gamma^\alpha\theta \qquad \theta_a \mapsto \theta'_a = \theta_a + \varepsilon_a$

Supermanifolds

Supermanifolds: the good, the bad and the ugly

two mathematical definitions of supermanifolds:

1. sets with a suitable topology *[DeWitt, Rogers]*

   $\mathbb{R}^{m|k} \to \mathbb{R}_L^{m|k} \simeq (\Lambda^\bullet \mathbb{R}^L)_0^m \times (\Lambda^\bullet \mathbb{R}^L)_1^k$

   arbitrarily large, possibly infinite, number L of Grassmann parameters needed ☹

2. indirect definition by defining functions on CSM *[Berezin, Leĭtes, Manin, ...]*

   finite number of Grassmann parameters sufficient ☺

Supermanifolds: the good

- smooth m-manifold $M \simeq$ *locally ringed space* (M, C^∞_M) with *structure sheaf*

$$C^\infty_M : M \supset U \mapsto C^\infty(U, \mathbb{R})$$

- smooth $m|k$-supermanifold $\mathcal{M} :=$ *locally superringed space* $\mathcal{M} = (M, \mathfrak{C})$ with structure sheaf

$$\mathfrak{C} : U \mapsto \mathfrak{C}(U) \simeq C^\infty(U, \mathbb{R}) \otimes_{\mathbb{R}} \Lambda^\bullet \mathbb{R}^k$$

$\rightarrow f \in \mathfrak{C}(M)$, i.e. "functions on $\mathcal{M}$" are locally (θ_a a basis of $\mathbb{R}^k$)

$$f \simeq \sum_{(i_1,\cdots,i_k)\in\mathbb{Z}_2^k} f_{i_1,\cdots,i_k}(x) \otimes_{\mathbb{R}} \theta_1^{i_1} \wedge \cdots \wedge \theta_k^{i_k} =: \sum f_{i_1,\cdots,i_k} \theta_1^{i_1} \cdots \theta_k^{i_k}$$

From spin manifolds to supermanifolds I

1. consider globally hyperbolic smooth manifold 4-dim $(M, g) \Rightarrow$ trivial Lorentz frame bundle $FM \simeq M \times SO_0(1,3)$

2. choose trivial spin structure $SM \simeq M \times Spin_0(1,3)$

3. construct Majorana spinor bundle DM

4. pick global frame E_a of DM, any section $f \in C^\infty(M, \Lambda^\bullet DM) \simeq C^\infty(M, \mathbb{R}) \otimes_\mathbb{R} \Lambda^\bullet \mathbb{R}^4$ can be written as

$$f = \sum_{(i_1,\cdots,i_4)\in\mathbb{Z}_4^2} f_{i_1,\cdots,i_4}(x)\ E_1^{i_1}(x) \wedge \cdots \wedge E_4^{i_4}(x) \simeq \sum f_{i_1,\cdots,i_f} \theta_1^{i_1} \cdots \theta_4^{i_4}$$

From spin manifolds to supermanifolds II

- $\Rightarrow \mathcal{M} = (M, \mathfrak{C})$ with

$$\mathfrak{C}(M) := C^\infty(M, \Lambda^\bullet DM) \simeq C^\infty(M, \mathbb{R}) \otimes_\mathbb{R} \Lambda^\bullet \mathbb{R}^k =: \mathfrak{C}_0$$

  defines a $4|4$ supermanifold

- we have "global Fermionic coordinates" θ_a (corresponding to a global vielbein of DM) and thus avoid sheaf-theoretic complications

- by construction the coefficients $f_{i_1,\cdots,i_4}$ of $f \in \mathfrak{C}_0$ transform under antisymmetrised products of the Majorana representation and can be interpreted as fields of different spin

- $\Rightarrow$ rigorous understanding of (in 3d)

$$\Phi = \phi + \overline{\psi}\theta + \frac{1}{2}F\theta^2$$

Differential Geometry on Supermanifolds

Vector fields on supermanifolds

- $\mathfrak{G}_0$ is a supercommutative superalgebra with a $\mathbb{Z}_2$-grading induced by the $\mathbb{Z}$-grading of $\Lambda^\bullet \mathbb{R}^k$
- vector fields on the supermanifold $\mathcal{M} = (M, \mathfrak{G})$ are derivations $X \in Der(\mathfrak{G}_0)$ of $\mathfrak{G}_0$, e.g. homogeneous derivation acting on $f, h \in \mathfrak{G}_0$, f homogeneous
$$X(fh) = X(f)h + (-1)^{|X||f|} fX(h)$$
- a ($\mathfrak{G}_0$-left supermodule) basis of $Der(\mathfrak{G}_0)$ is given by $(e_0, \cdots, e_4, \partial^1, \cdots, \partial^4)$ where e_α vierbein and $\partial^a = \frac{\partial}{\partial \theta_a}$

Forms and differentials on supermanifolds

- $\Omega(\mathfrak{G}_0)$ dual of $Der(\mathfrak{G}_0)$ with basis $(e^\alpha, d\theta_a)$

$$\langle e_\alpha, e^\beta \rangle = \delta_\alpha^\beta \quad \langle \partial^a, e^\alpha \rangle = 0 \quad \langle e_\alpha, d\theta_a \rangle = 0 \qquad \langle \partial^b, d\theta_a \rangle = \delta_a^b$$

- differential $d : \mathfrak{G}_0 \to \Omega(\mathfrak{G}_0)$ defined by $\langle X, df \rangle := X(f)$ for all $X \in Der(\mathfrak{G}_0)$, $f \in \mathfrak{G}_0$

- higher forms + extension of d can be introduced

Superdiffeomorphisms I

- given two $m|k$ supermanifolds $\mathcal{M}_1$ and $\mathcal{M}_2$, "superdiffeomorphisms" $\chi \in \mathrm{Hom}_{\mathsf{SMan}}(\mathcal{M}_1, \mathcal{M}_2)$ can be abstractly defined as morphisms of locally superringed spaces, but a general explicit characterisation seems unavailable to date
- however, just as $\mathrm{Hom}_{\mathsf{Man}}(M_1, M_2) \simeq \mathrm{Hom}_{\mathsf{Alg}}(C^\infty(M_2), C^\infty(M_1)) \simeq \{\text{smooth functions } y^1, \cdots, y^m \in C^\infty(M_1) \text{ s.t. } (y^1(M_1), \cdots, y^m(M_2)) \subset M_2\}$ for $M_i \subset \mathbb{R}^m$ ...
- ... one can show $\mathrm{Hom}_{\mathsf{SMan}}(\mathcal{M}_1, \mathcal{M}_2) \simeq \mathrm{Hom}_{\mathsf{SAlg}}(\mathfrak{G}(M_2), \mathfrak{G}(M_1)) \simeq \{\text{odd and even functions } y^1, \cdots, y^m, \xi_1, \cdots, \xi_k \cdots, \in \mathfrak{G}(M_1) \text{ s.t. } (\beta(y^1(M_1)), \cdots, \beta(y^m(M_1))) \subset M_2\}$ for $\mathcal{M}_i = (M_i, \mathfrak{G})$, $M_i \subset \mathbb{R}^m$

Superdiffeomorphisms II

- given $y^\alpha, \xi_a \in \mathfrak{G}(M_1)$, $\psi_{y,\xi} \in \mathrm{Hom}_{\mathrm{SAlg}}(\mathfrak{G}(M_2), \mathfrak{G}(M_1))$ is defined by a "pullback", e.g. (in 3d)

$$y^\alpha = a^\alpha(x) + b^\alpha(x)\theta^2 \qquad \xi_a = m_a^b(x)\theta_b$$

$$\psi_{y,\xi}\left(\phi + \overline{\psi}\theta + \frac{1}{2}F\theta^2\right) := \phi(a^\alpha) + \partial_\beta\phi(a^\alpha)b^\beta\theta^2 + \cdots + \psi^a m_a^b\theta_b + \cdots$$

- but $\psi_{y,\xi}$ must preserve parity $\Rightarrow$ only parity preserving y^α, ξ_b allowed, SUSY supertranslations $y^\alpha = x^\alpha + \bar{\varepsilon}\gamma^\alpha\theta$ disallowed unless one introduces "flesh"!

- however, infinitesimal SUSY supertranslations are available as odd derivations on a fixed supermanifold

$$X = \bar{\varepsilon}\gamma^\alpha\theta e_\alpha + \varepsilon^a\partial_a$$

Superconnection and supervielbein I

- on ordinary manifolds, we can combine a connection ω and dual vielbein e^α into a "Cartan connection"

$$\Gamma := \omega + e = \omega^{\beta\gamma} L_{\beta\gamma} + e^\alpha P_\alpha \in \Omega(M) \otimes_{\mathbb{R}} \mathfrak{poin}(1,3)$$

$L_{\beta\gamma}, P_\alpha$ generators of $\mathfrak{poin}(1,3)$, Lie algebra of $Poin(1,3) := Spin_0(1,3) \ltimes \mathbb{R}^4$

- Γ can be interpreted as induced by a connection on a $Poin(1,3)$-bundle after reducing the latter to a $Spin_0(1,3)$-bundle
- Levi-Civita connection ω can be specified by a suitable torsion constraint

Superconnection and supervielbein II

- on the supermanifold $\mathcal{M} = (M, \mathfrak{G}_0)$, we want to consider a "super Cartan connection"

$$\widehat{\Gamma} = \widehat{\omega} + \widehat{e} + \widetilde{e} = \widehat{\omega}^{\beta\gamma} L_{\beta\gamma} + \widehat{e}^{\alpha} P_{\alpha} + \widetilde{e}^{a} Q_a \in \Omega(\mathcal{G}_0) \otimes_{\mathbb{R}} \mathfrak{suppoin}(1,2)$$

coming from a reduction $Suppoin(1,3) \to Spin_0(1,3)$

- $\mathfrak{suppoin}(1,3)$ super Lie algebra generated by even $L_{\beta\gamma}, P_\alpha$, odd $Q_a^{L/R}$

$$[L_{\alpha\beta}, Q_a^{L/R}] = ([\gamma_\alpha, \gamma_\beta])_a{}^b \, Q_b^{L/R} \qquad [P_\alpha, Q_b^{L/R}] = 0$$

$$[Q_a^L, Q_b^R] = (\pi_L \gamma^\alpha)_{ab} P_\alpha \qquad [Q_a^L, Q_b^L] = 0 \qquad \pi_{L/R} = \frac{1}{2}\left(1 \mp i\gamma^5\right)$$

Superconnection and supervielbein III

- textbooks: suitable "supertorsion constraints" + "superdiffeomorphism gauge fixing": $\widehat{\Gamma}$ is specified in terms of $(e^\alpha, \psi^a, a, b^\alpha)$, we choose "metric backgrounds" $(e^\alpha, 0, 0, 0) \Rightarrow$

$$\widehat{e}^\alpha = e^\alpha - d\theta_a (\gamma^\alpha)^{ab} \theta_b \qquad \widetilde{e}^a = d\theta^a + e^\alpha g_\alpha^{ab} \theta_b$$

$$\widehat{\omega} = \omega = \text{Levi-Civita connection} \qquad g_\alpha^{ab} \text{ specified in terms of } \omega$$

- define derivations $\widehat{e}_\alpha$, $\widetilde{e}_a \in Der(\mathfrak{G}_0)$ as inverses of $\widehat{e}^\alpha$, $\widetilde{e}^a$
- $E := \widehat{e} + \widetilde{e} = E^A (Q \oplus P)_A \Rightarrow \text{sdet}\left(E^A\right) = \det(e^\alpha)$

Category of CSM for QFT

- $\Rightarrow$ "best" category SLoc of backgrounds for locally supercovariant QFT so far:

$$\mathrm{Obj}_{\mathrm{SLoc}} = \{\mathfrak{M} = (\mathcal{M}, E) \,|\, \mathcal{M} = (M, \mathfrak{G}_0), (M, e^\alpha) \text{ glob. hyp.}, M \subset \mathbb{R}^4\}$$

$$\mathrm{Hom}_{\mathrm{SLoc}}(\mathfrak{M}_1, \mathfrak{M}_2) =$$
$$\{E - \text{preserving SMan morphisms which induce}$$
$$\text{c.c. isom. embed. } (M_1, e_1^\alpha) \to (M_2, e_2^\alpha)\}$$

The free Wess-Zumino model as a QFT on CSM

The free Wess-Zumino model as a QFT on CSM

- plan: construct the free quantum Wess-Zumino model as an algebraic QFT on CSM (first: arbitrary $4|4$ "spin"-CSM)
- a linear QFT on CST is specified by
  1. a vector bundle with non-degenerate bilinear form
  2. an equation of motion

Chiral superfields

- chiral coordinates $\theta_a^{L/R} := \left(\pi^{L/R}\right)_a^{\;b}\theta_b \qquad \theta^2_{L/R} := \overline{\theta^{L/R}}\theta^{L/R}$
- (left) chiral superfields $\Phi \in \mathfrak{C}_L \subset \mathfrak{G}_0^{\mathbb{C}} := \mathfrak{G}_0 \otimes_{\mathbb{R}} \mathbb{C}$

$$D_a^R\Phi := \langle \pi^R\widetilde{e}_a, d\Phi\rangle = 0$$

$$\Rightarrow \Phi = \phi + \sqrt{2}\,\overline{\psi}\theta_L + F\theta_L^2 + \overline{\theta_R}\partial\!\!\!/\phi\theta_L + \frac{1}{4}\Box\phi\theta_L^2\theta_R^2 - \frac{1}{\sqrt{2}}\overline{\theta_R}\nabla\!\!\!\!/\,\psi\theta_L^2$$

- $\Phi \in \mathfrak{G}_L \Leftrightarrow \Phi^* \in \mathfrak{G}_R$
- define $\mathfrak{G}^{\oplus} \ni \Phi^{\oplus} = (\Phi, \Phi^*)^T$, $\Phi \in \mathfrak{G}_L$

Test superfunctions and bilinear form

- "chiral supertestfunctions" $\mathfrak{G}_0^{\oplus} \ni F = (f, f^*)^T$, f has smooth and compactly supported expansion coefficients

- non-degenerate bilinear form on $\mathfrak{G}_0^{\oplus}$

$$\langle F_1, F_2 \rangle := \Re \int_M dx \int d\theta_L^2 \; \mathrm{sdet}(E^A) f_1 f_2 = \Re \int_M d\mathrm{Vol} \int d\theta_L^2 \; f_1 f_2$$

$$\int d\theta_L^2 \theta_L^2 := 1 \,, \qquad \text{integrals of linear } \theta_L \text{ polynomials}{=}0$$

Equation of motion

- define "chiral projector" $P_L : \mathfrak{G}^{\mathbb{C}} \to \mathfrak{G}^{\mathbb{C}}$ by

$$P_L\Phi := -\frac{1}{4}C^{ab}\langle \pi^L\widetilde{e}_b, d(D_a^L\Phi)+\omega_a{}^c D_c^L\Phi\rangle + \frac{1}{6}R\theta_R^2\Phi = \left(-\frac{1}{4}D_L^a D_a^L + \frac{1}{6}R\theta_R^2\right)\Phi$$

- $\Rightarrow P_L : \mathfrak{G}_L \mapsto \mathfrak{G}_R$ and for $\Phi \in \mathfrak{G}_L$

$$P_L\Phi = F - \sqrt{2}\,\overline{\theta}_R \nabla\!\!\!/\psi + \left(\Box + \frac{1}{6}R\right)\phi\theta_R^2 + \frac{1}{\sqrt{2}}\overline{\nabla\!\!\!/^2\psi}\theta_L\theta_R^2 + \frac{1}{4}\Box F\theta_L^2\theta_R^2 + \overline{\theta}_R \partial\!\!\!/ F\theta_L$$

- define (massless) chiral superwave operator $P : \mathfrak{G}^{\oplus} \to \mathfrak{G}^{\oplus} \ni \Phi^{\oplus}$: by

$$P\Phi^{\oplus} := P\begin{pmatrix}\Phi \\ \Phi^*\end{pmatrix} := \begin{pmatrix} 0 & P_R \\ P_L & 0\end{pmatrix}\begin{pmatrix}\Phi \\ \Phi^*\end{pmatrix}$$

- $\langle Pf_1^{\oplus}, f_2^{\oplus}\rangle = \langle f_1^{\oplus}, Pf_2^{\oplus}\rangle$ and P has unique advanced/retarded Green's operators $G_P^{\pm}$, $G_P := G_P^{-} - G_P^{R}$

Quantization I

- field algebra $\mathcal{A}(\mathcal{M})$ of the quantum Wess-Zumino field on $\mathcal{M}$ is free tensor algebra generated by $\underline{1}$, $\Phi(f)$ (the quantizations of the functionals $\langle \Phi^{\oplus}, f^{\oplus} \rangle$) and relations

$$\Phi(P_R f^*) = 0$$

$$\Phi(f_1)\Phi(f_2) - (-1)^{|f_1||f_2|}\Phi(f_2)\Phi(f_1) = i\langle f_1^{\oplus}, G_P f_2^{\oplus} \rangle \underline{1}$$

for f_i homogeneous

Quantization II

- $\Rightarrow$ the canonical supercommutation relations combine the CCR for ϕ and the CAR for ψ in a nice way!

$$\phi(h) \simeq \Phi(f) \qquad f = (h - ih)\theta_L^2$$

$$\psi(p) \simeq \Phi(f) \qquad f = \sqrt{2}\overline{p}\theta_L - \frac{1}{\sqrt{2}}\overline{\theta}_R \nabla\!\!\!\!/\, p\theta_L^2$$

- quantization can be formulated in terms of a functor $\mathcal{A} : \mathrm{SLoc} \to \mathrm{Alg}$

Thanks a lot for your attention!