The DFR-Algebra for Poisson Vector Bundles

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The DFR Model

History

- The DFR model is a model for "quantum space-time", proposed in 1995 by Doplicher, Fredenhagen and Roberts (DFR).
- The authors construct a special C^*-algebra to provide a model for space-time in which the localization of events can no longer be performed with arbitrary precision, but which has the usual Minkowski space as its classical limit.

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Reinterpreting the construction of the DFR algebra

- The construction of the DFR algebra can be reformulated in the following terms:
  - One starts with the choice of a symplectic form σ on Minkowski space and considers the corresponding finite-dimensional nilpotent Lie algebra: the Heisenberg Lie algebra.

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- Next, this representation is used to generate a C^*-algebra, the Heisenberg C^*-algebra, whose product is determined by Weyl quantization and can be expressed through the Weyl-Moyal star product.
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- Assuming the symplectic form to vary over the orbit of some fixed representative produces not just a single Heisenberg C^*-algebra but an entire bundle of C^*-algebras over that orbit, with the Heisenberg C^*-algebra as typical fiber.
- The continuous sections of that bundle vanishing at infinity define a C^*-algebra which carries a natural action of the Lorentz group and which is also a C^*-module over the "scalar" C^*-algebra of continuous functions on the orbit vanishing at infinity.

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Generalizing the DFR Model

Generalizing the construction of the DFR algebra

- Our main goal in this work has been to generalize the above construction to classical space-time manifolds other than Minkowski space: this requires removing the Lorentz group.
- Almost as a by-product, we have also been able to eliminate the hypothesis that the form σ should be nondegenerate.

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The Heisenberg Lie Algebra and Lie Group

The Heisenberg Lie algebra of a Poisson vector space

- Given a Poisson vector space V with Poisson bivector σ, we define the associated Heisenberg algebra or, more precisely, *Heisenberg Lie algebra* $\mathfrak{h}_\sigma$; as a vector space, $\mathfrak{h}_\sigma = V^* \oplus \mathbb{R}$, and the commutator is given by

$$[(\xi,\lambda),(\eta,\mu)] \; = \; (0\,,\,\sigma(\xi,\eta))$$
$$\text{for} \;\; \xi,\eta \in V^*\,,\; \lambda,\mu \in \mathbb{R}\;.$$

The Heisenberg Lie Algebra and Lie Group

The Heisenberg Lie group of a Poisson vector space

- Exponentiating, we obtain the *Heisenberg Lie group* H_σ: as a manifold, $H_\sigma = V^* \times \mathbb{R}$, and the product (written additively) is given by

$$(\xi, \lambda)\,(\eta, \mu) \;=\; \big(\xi + \eta\,,\, \lambda + \mu - \tfrac{1}{2}\,\sigma(\xi, \eta)\big)$$
$$\text{for}\quad \xi, \eta \in V^*\,,\, \lambda, \mu \in \mathbb{R}\;.$$

The Schrödinger Representation

The Schrödinger representation of the Heisenberg Lie algebra

- Using the Schrödinger representation of the usual canonical commutation relations, we can define a strongly continuous unitary representation of the Heisenberg Lie group which will be denoted by π_σ. After abbreviating $\pi_\sigma(\xi, 0)$ to $\pi_\sigma(\xi)$, satisfies

$$\pi_\sigma(\xi)\,\pi_\sigma(\eta) \;=\; e^{-\frac{i}{2}\sigma(\xi,\eta)}\,\pi_\sigma(\xi+\eta)\;.$$

The Heisenberg C^*-Algebra

The Weyl quantization map

- This construction can be further extended to the C^*-algebra setting using *Weyl quantization*, which in the context to be considered here will be regarded as a continuous linear map

$$\begin{array}{rccc} W_\sigma : & \mathcal{S}(V) & \longrightarrow & B(L^2(\mathbb{R}^{n-r})) \\ & f & \longmapsto & W_\sigma f \end{array}$$

  where $\mathcal{S}(V)$ denotes the space of Schwartz test functions on V and $B(L^2(\mathbb{R}^{n-r}))$ the space of bounded linear operators on the Hilbert space $L^2(\mathbb{R}^{n-r})$, constructed as follows:

The Heisenberg C^*-Algebra

The Weyl quantization map

- For $f \in \mathcal{S}(V)$, set

$$W_\sigma f \;=\; \int_{V^*} d\xi\; \check{f}(\xi)\, \pi_\sigma(\xi)\;,$$

where $\check{f} \in \mathcal{S}(V^*)$ is the inverse Fourier transform of f,

$$\check{f}(\xi) \;=\; \frac{1}{(2\pi)^n} \int_V dx\; f(x)\, e^{-i\langle \xi, x\rangle}\;,$$

The Heisenberg C^*-Algebra

The Weyl-Moyal star product

- It can be shown that W_σ is faithful, and an explicit calculation gives

$$W_\sigma f \; W_\sigma g \;=\; W_\sigma(f \star_\sigma g) \qquad \text{for} \quad f, g \in \mathcal{S}(V) \;,$$

where $\star_\sigma$ denotes the *Weyl-Moyal star product*, given by:

$$(f \star_\sigma g)(x) \;=\; \int_{V^*} d\xi \; e^{i\langle \xi, x \rangle} \int_{V^*} d\eta \; \check{f}(\eta)\, \check{g}(\xi - \eta) \; e^{-\frac{i}{2}\sigma(\xi,\eta)}$$

$$\text{for} \quad f, g \in \mathcal{S}(V) \;.$$

The Heisenberg C^*-Algebra

The Heisenberg-Schwartz algebra

- The Weyl-Moyal star product, together with the standard involution of pointwise complex conjugation, turns $\mathcal{S}(V)$ into a Fréchet $*$-algebra, which we call the *Heisenberg-Schwartz algebra* and denote by $\mathcal{S}_\sigma$.
- Obviously, $\mathcal{S}_\sigma$ also carries a C^*-norm, namely the one given by the pull back of the operator norm by the Weyl quantization W_σ: its completion in this norm is then a C^*-algebra, which we denote by $\mathcal{E}_\sigma$.

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- The C^*-algebra $\mathcal{E}_\sigma$ is what we propose to call the *Heisenberg C^*-algebra*.
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The Heisenberg Lie Algebroid and Lie Groupoid

Poisson vector bundles

- In what follows, we assume that E is a Poisson vector bundle, i.e., areal vector bundle of fiber dimension n, say, over a manifold M, equipped with a fixed bivector field σ.
- Then it is clear that we can apply all the constructions of the previous section to each fiber. The question to be addressed in this section is how the results can be glued together along the base manifold M and to describe the resulting global objects.

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The Heisenberg Lie Algebroid and Lie Groupoid

The Heisenberg Lie algebroid of a Poisson vector bundle

- Starting with the collection of Heisenberg Lie algebras $\mathfrak{h}_{\sigma(m)}$ ($m \in M$), we fit them together into a real vector bundle over M, which is simply the direct sum of the dual vector bundle E^* and the trivial line bundle $M \times \mathbb{R}$.
- The commutator, defined fiberwise in terms of the bivector σ_m over each fiber E_m, turns this vector bundle into a Lie algebroid which we shall call the *Heisenberg Lie algebroid* associated to (E, σ) and denote by $\mathfrak{h}(E, \sigma)$.

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- Similarly, considering the collection of Heisenberg groups $H_{\sigma(m)}$ $(m \in M)$, we can fit them together into a fiber bundle, which is simply the fiber product of the dual vector bundle E^* and the trivial line bundle $M \times \mathbb{R}$.
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Spaces of sections

- Of course, spaces of sections of $\mathfrak{h}(E,\sigma)$ and of $H(E,\sigma)$ (with certain decay conditions) will then form (infinite-dimensional) Lie algebras and Lie groups respectively, we are also free to make choices of decay conditions.

The DFR-Algebra

Fibrations of Heisenberg C^*-algebras: the idea

- The same strategy can be applied to the collection of Heisenberg C^*-algebras $\mathcal{E}_{\sigma(m)}$ $(m \in M)$.
- However, the details are somewhat intricate since here, the fibers are (infinite-dimensional) C^*-algebras whose structure may depend on the base point in a discontinuous way, since the rank of σ is allowed to jump, and this jeopardizes local triviality.

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- The space of continuous sections of such fibration that vanish at infinity, $\mathscr{E}(E,\sigma)$, is a C^*-algebra which we propose to call the *DFR-Algebra* associated to the original Poisson vector bundle.

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The Heisenberg-Schwartz bundle

- To achieve this, we define the Schwartz bundle associated to (E, σ) whese fibers are just the Schwartz space over each fiber of the original bundle. It can be show that this defines a infinite dimentional smooth vector bundle over M.
- We consider then the fiberwise Weyl-Moyal star product, defined in terms of the bivector σ_m over each fiber E_m, turning this vector bundle into a fibration of Fréchet $*$-algebras over M which we shall call the *Heisenberg-Schwartz Fibration* associated to (E, σ) and denote by $\mathcal{S}(E, \sigma)$.

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The DFR-Algebra

The algebra of sections of the Heisenberg-Schwartz bundle

- The space $C_0(\mathcal{S}(E,\sigma) \to M)$ of continuous sections of $\mathcal{S}(E,\sigma)$ vanishing at infinity is again a Fréchet $*$-algebra which we shall denote by $\mathscr{S}(E,\sigma)$.
- (Of course, we might also consider smooth sections with appropriate decay properties at infinity: whether this will give useful additional information is still an open question.)

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The algebra of sections of the Heisenberg-Schwartz bundle

- Besides that, $\mathscr{S}(E,\sigma)$ is also a module over the “scalar” function algebra $C_b(M)$ with respect to the obvious pointwise multiplication of sections by functions.
- Finally, for every point m in M, we have the evaluation map, at m, which is a surjective continuous $*$-algebra homomorphism

$$\begin{array}{rccc} \delta_m : & \mathscr{S}(E,\sigma) & \longrightarrow & \mathcal{S}_{\sigma(m)} \\ & \varphi & \longmapsto & \varphi(m) \end{array}$$

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- Now, for every point m in M, composing the evaluation map δ_m at m with the Weyl quantization map $W_{\sigma(m)}$ yields a $*$-representation $W_m = W_{\sigma(m)} \circ \delta_m$ of $\mathscr{S}(E,\sigma)$ by bounded operators on a certain Hilbert space, and the family of all these $*$-representations is separating.
- This proves the existence of a C^*-norm on $\mathscr{S}(E,\sigma)$, explicitly given by:

$$\|\varphi\|_{\mathscr{E}} = \sup_{m\in M} \|\varphi(m)\|_{\mathcal{E}_{\sigma(m)}} = \sup_{m\in M} \|W_{\sigma(m)}(\delta_m\varphi)\| \quad \text{for} \quad \varphi \in \mathscr{S}(E,\sigma) .$$

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- The completion of $\mathscr{S}(E, \sigma)$ in this norm is then a C^*-algebra, which we denote by $\mathscr{E}(E, \sigma)$.
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The DFR-algebras as modules

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The DFR-Algebra

Compatibility between module structure and evaluation maps

- In all cases, the module structure and the evaluation maps are related by the obvious formula

$$\delta_m(f\varphi) \;=\; f(m)\,\delta_m(\varphi)$$

$$\text{for}\quad f \in C_b(M) \quad \text{and} \quad \varphi \in \mathscr{S}(E,\sigma) \text{ or } \mathscr{E}(E,\sigma).$$

The DFR-Algebra

The DFR-Algebra an algebra of sections

- Due to this compatibility contiditon, it is clear that the module scructure is continuous, in the sence that:

$$\|f\varphi\|_{\mathscr{E}} \leqslant \|f\|_\infty \|\varphi\|_{\mathscr{E}}$$

$$\text{for} \quad f \in C_b(M), \quad \varphi \in \mathscr{E}(E,\sigma) \ .$$

- This implies the module structure defines an homomorphism between the algebra $C_b(M)$ and the center of the multiplier algebra of $\mathscr{E}(E,\sigma)$, so that the later is what is called in the literature a $C_0(M)$-algebra.

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- A general result for such algebras allows us to obtain a topology on the space

$$\mathcal{E}(E,\sigma) = \dot{\bigcup_{m \in M}} \mathcal{E}_{\sigma(m)}$$

that turns it into a Fibration of C^*-algebras over M and such that the DFR-algebra is isomorphic to the algebra of sections $C_0(\mathcal{E}(E,\sigma) \to M)$.

Group Actions – The Homogeneous Case

Homogeneous Vector Bundles

- An important special and also simpler case of the construction outlined above occurs when the underlying manifold M and Poisson vector bundle (E, σ) are homogeneous.
- This means that there is a Lie group G which acts properly both on M and on E, transitively on M and linearly on the fibers of E, in such a way that σ is G-invariant.

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- In this case, fixing any point m_0 in M and denoting by G_0 its stability group, by E_0 the fiber of E over m_0 and by σ_0 the value of the bivector field σ at m_0, we can identify
  - M with the homogeneous space G/G_0,

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