Exact solution of a non-local four-dimensional quantum field theory

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The Kontsevich model

- 2D quantum gravity can be formulated as a one-matrix model with partition function

$$\mathcal{Z} = \int dM \; \exp\Big(-\mathcal{N}\sum_n t_n \,\mathrm{tr}(M^n)\Big)\;, \qquad M = M^* \in M_{\mathcal{N}}(\mathbb{C})$$

- For $\mathcal{N} \to \infty$, this series in (t_n) can be expressed in terms of the τ-function for the Korteweg-de Vries (KdV) hierarchy.
- Topological gravity leads to another series in (t_n) with coefficients given by intersection numbers of complex curves.
- Witten conjectured in 1990 that both series are the same.

- Kontsevich computed in 1992 the intersection numbers in terms of weighted sums over ribbon graphs.
- He proved these graphs to be generated from the Airy function matrix model (Kontsevich model)

$$\mathcal{Z}[E] = \frac{\int dM \;\exp\big(-\frac{1}{2}\mathrm{tr}(EM^2) + \frac{\mathrm{i}}{6}\mathrm{tr}(M^3)\big)}{\int dM \;\exp\big(-\frac{1}{2}\mathrm{tr}(EM^2)\big)}\,, \qquad M{=}M^*{\in}M_{\mathcal{N}}(\mathbb{C})$$

for $E = E^* > 0$ and $t_n = (2n{-}1)!!\mathrm{tr}(E^{-(2n-1)})$.

- Limit $\mathcal{N} \to \infty$ of $\mathcal{Z}[E]$ gives the KdV evolution equation, thus proving Witten's conjecture.

A matrix model inspired by noncommutative QFT

- The simplest QFT on a 4D noncommutative manifold can be written as a matrix model

$$\mathcal{Z}[E,J,\lambda] = \frac{\int dM\ \exp\big(-\mathrm{tr}(EM^2)+\mathrm{tr}(JM)-\frac{\lambda}{4}\mathrm{tr}(M^4)\big)}{\int dM\ \exp\big(-\mathrm{tr}(EM^2)-\frac{\lambda}{4}\mathrm{tr}(M^4)\big)},$$

  where $E = E^* \in M_{\mathcal{N}}(\mathbb{C})$ is the 4D Laplacian, $\lambda \geq 0$ and $J \in M_{\mathcal{N}}(\mathbb{C})$ generates correlation functions.
- We achieve the exact solution of $\mathcal{Z}[E,J,\lambda]$ for $\mathcal{N}\to\infty$ and after renormalisation of E,λ.
- This defines a QFT toy model in four dimensions, which is non-trivial with coupling constant $0 \leq \lambda \leq 64\pi$.

We have no idea what mathematical structure made this possible.

Outline of the renormalisation

- Expanding $\exp(\mathrm{tr}(-\frac{\lambda}{4}M^4))$ perturbatively gives infinitely many divergent matrix integrals (the same as for ϕ_4^4).
- Renormalisation is achieved in two steps: thermodynamic limit and continuum limit.

1. First $\lambda \mapsto \mathcal{N}^2\lambda$ and $E \mapsto \mathcal{N}^2 E_{\mathcal{N}}$ are made $\mathcal{N}$-dependent. Double-scaling limit $\mathcal{N} \to \infty$ corresponds to infinite-volume limit in position space.
   - The spectrum of E becomes continuous but with UV-cutoff, $[0, \Lambda^2]$.
   - Leads to $\sum_{p=0}^{\mathcal{N}} f(p) \mapsto \int_0^{\Lambda^2} d\mu(\rho) f(\rho)$, with $d\mu(\rho)$ the spectral density of E.

2. Integrals for 2- and 4-point functions diverge for $\Lambda \to \infty$.
   - We introduce a Λ-dependence in E corresponding to mass and wavefunction renormalisation.
   - Cancellation of the Λ-divergence in the 2-point function also cancels divergence in 4-point function (i.e. $\beta = 0$).

We would have been happy just proving that this prescription constructs the model non-perturbatively for some $\lambda > 0$.

- But much more is achieved: We can compute any renormalised correlation function exactly in $0 \leq \lambda \leq 64\pi$.
- This involves a new special function $G^\lambda : \mathbb{R}_+ \to [0, 1]$.
- Key ingredients are Schwinger-Dyson techniques and the theory of Carleman type singular integral equations.

There are a few gaps which all seem closable.

Field-theoretical matrix models

- classical scalar field $\phi \in \mathcal{C}_0(\mathbb{R}^d) \subset \mathcal{B}(H)$, with $\frac{m}{2}\int_{\mathbb{R}^d} dx\ \phi^2(x)$
- translates to $\mathrm{tr}(\phi^2) < \infty$, i.e. nc scalar field is Hilbert-Schmidt compact operator on Hilbert space $H = L^2(I,\mu)$
- realise as integral kernel operators: $M = (M_{ab}) \in L^2(I{\times}I, \mu{\times}\mu)$
  - product: $(MN)_{ab} = \int_I d\mu(c)\ M_{ac}M_{cb}$
  - trace: $\mathrm{tr}(M) = \int_I d\mu(a)\ M_{aa}$
  - adjoint: $(M^*)_{ab} = \overline{M_{ba}}$
- action = non-linear functional S for $\phi = \phi^*$:

$$S[\phi] = \mathrm{tr}(E\phi^2) + V[\phi]\ , \qquad V[\phi] = \mathrm{tr}(P[\phi])$$

E – unbounded positive selfadjoint op. with compact resolvent,
$P[\phi]$ – polynomial in ϕ with scalar coefficients

- partition function $\mathcal{Z}[J] = \int \mathcal{D}\phi\ \exp(-S[\phi] + \mathrm{tr}(\phi J))$

Ward identity

- Unitary transformation $\phi \mapsto U\phi U^*$ leads to Ward identity

$$0 = \int \mathcal{D}\phi \left[E\phi\phi - \phi\phi E - J\phi + \phi J\right] \exp(-S[\phi] + \mathrm{tr}(\phi J))$$

that describes how E, J break the invariance of the action.

...choose E (but not J) diagonal, use $\phi_{ab} = \frac{\partial}{\partial J_{ba}}$:

Proposition [Disertori-Gurau-Magnen-Rivasseau, 2006]

The partition function $\mathcal{Z}[J]$ of the matrix model defined by the external matrix E satisfies the $|I| \times |I|$ Ward identities

$$0 = \sum_{n \in I} \Big((E_a - E_p) \frac{\partial^2 \mathcal{Z}}{\partial J_{an} \partial J_{np}} + J_{pn} \frac{\partial \mathcal{Z}}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}}{\partial J_{np}} \Big)$$

For E of compact resolvent we can always assume that $m \mapsto E_m > 0$ is injective!

Topological expansion

- Connected Feynman graphs in matrix models are ribbon graphs.
- Viewed as simplicial complexes, they encode the topology (B,g) of a genus-g Riemann surface with B boundary components (or punctures, marked points, holes, faces).
- The k^{th} boundary component carries a cycle $J^{N_k}_{p_1\dots p_{N_k}} := \prod_{j=1}^{N_k} J_{p_j p_{j+1}}$ of N_k external sources, $N_k + 1 \equiv 1$.
- We expand $\mathcal{W}[J] = \sum \frac{1}{S} G_{|p_1\dots p_{N_1}|\dots|q_1\dots q_{N_B}|} J^{N_1}_{p_1\dots p_{N_1}} \cdots J^{N_B}_{q_1\dots q_{N_B}}$ according to the cycle structure.

The cycle structure determines the kernel of $(E_a - E_p)$ when applied to $\sum_{n\in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an}\partial J_{np}}$:

Theorem (Ward identity for injective E)

$$\sum_{n\in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an}\partial J_{np}}$$

$$= \delta_{ap}\Big\{ \sum_{(K)} \frac{J_{P_1}\cdots J_{P_K}}{S_K} \Big(\sum_{n\in I} G_{|an|P_1|\dots|P_K|} + G_{|a|a|P_1|\dots|P_K|}$$

$$+ \sum_{r\geq 1} \sum_{q_1,\dots q_r\in I} G_{|q_1 a q_1 \dots q_r|P_1|\dots|P_K|} J^r_{q_1\dots q_r} \Big)$$

$$+ \sum_{(K),(K')} \frac{J_{P_1}\cdots J_{P_K} J_{Q_1}\cdots J_{Q_{K'}}}{S_K S_{K'}} G_{|a|P_1|\dots|P_K|} G_{|a|Q_1|\dots|Q_{K'}|} \Big\} \mathcal{Z}[J]$$

$$-\frac{1}{E_a - E_p} \sum_{n\in I} \Big(J_{pn} \frac{\partial \mathcal{Z}[J]}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}[J]}{\partial J_{np}} \Big)$$

This formula lets the usually infinite tower of Schwinger-Dyson equations collapse:

Schwinger-Dyson equations (for $V[\phi] = \frac{\lambda_4}{4}\mathrm{tr}(\phi^4)$)

further expansion of connected functions $G_{\dots} = \sum_{g=0}^{\infty} G^{(g)}_{\dots}$ into components of equal genus g leads to a short system of Schwinger-Dyson equations:

1. A closed non-linear equation for $G^{(0)}_{ab}$ (planar+regular):

$$G^{(0)}_{|ab|} = \frac{1}{E_a + E_b} - \frac{\lambda_4}{E_a + E_b} \sum_{p \in I} \Big(G^{(0)}_{|ab|} G^{(0)}_{|ap|} - \frac{G^{(0)}_{|pb|} - G^{(0)}_{|ab|}}{E_p - E_a} \Big)$$

2. For every other $G^{(g)}_{a_1 \dots a_N}$ an equation which only depends on

- $G^{(g)}_{a_1 \dots a_k}$ for $k \le N$,
- $G^{(h)}_{a_1 \dots a_k}$ with $h < g$ and $k \le N+2$;

this dependence is linear in the top degree (N, g)

Some $G_{\dots}$ need renormalisation of E, ϕ, and λ_n!

ϕ^4_4 on Moyal space with harmonic propagation

ϕ^4-theory on 4D-Moyal space w/ harmonic oscillator potential

$$S[\phi] = \int d^4x \Big(\frac{Z}{2}\phi \star \big(-\Delta + \Omega^2 (2\Theta^{-1}x)^2 + \mu^2_{bare}\big)\phi + \frac{\lambda Z^2}{4}\phi\star\phi\star\phi\star\phi\Big)(x)$$

- renormalisable as formal power series in λ [Grosse-W., 2004] (renormalisation of $\mu^2_{bare}, \lambda, Z \in \mathbb{R}_+$ and $\Omega \in [0,1]$) means: well-defined perturbative quantum field theory
- Langmann-Szabo duality (2002): theories at Ω and $\Omega^* = \frac{1}{\Omega}$ are the same; self-dual case $\Omega = 1$ is matrix model
- β-function vanishes to all orders in λ for $\Omega = 1$ [Disertori-Gurau-Magnen-Rivasseau, 2006] means: almost scale-invariant

Is the self-dual (critical) model integrable?

Matrix basis and thermodynamic limit

The Moyal algebra has a matrix basis [Gracia-Bondía+Várilly, 1988] in which the previous action becomes for $\Omega = 1$

$$S[\phi] = \sum_{\underline{m},\underline{n}\in\mathbb{N}^2_{\mathcal{N}}} E_{\underline{m}}\, \phi_{\underline{m}\underline{n}}\phi_{\underline{n}\underline{m}} + \frac{(2\pi\theta)^2 Z^2 \lambda}{4} \sum_{\underline{m},\underline{n},\underline{k},\underline{l}\in\mathbb{N}^2_{\mathcal{N}}} \phi_{\underline{m}\underline{n}}\phi_{\underline{n}\underline{k}}\phi_{\underline{k}\underline{l}}\phi_{\underline{l}\underline{m}}$$

$$E_{\underline{m}} = (2\pi\theta)^2 Z \left(\frac{4}{\theta}|\underline{m}| + \frac{\mu^2_{bare}}{2} \right), \qquad |\underline{m}| := \underline{m}_1 + \underline{m}_2 \leq \mathcal{N}$$

- $(2\pi\theta)^2$ is for $\Omega = 1$ the volume of the noncommutative manifold which is sent to ∞ in the thermodynamic limit.
- We do this in the double-scaling limit $\frac{4\mathcal{N}}{\theta} = \Lambda^2\mu^2 = \mathrm{const}$
- Matrix indices become continuous $\frac{4}{\theta}|\underline{p}| \mapsto \mu^2 p$ with $p \in [0, \Lambda^2]$.

- θ drops out in SD-equations (appearing via $E, \lambda_4 = (2\pi\theta)^2 Z^2\lambda$) if the following function is θ-independent:
$$G^g_{P_1|\dots|P_B} := \mu^{-N}(2\pi\theta\mu^2)^{2N-4+2B+4g} G^{(g)}_{|\underline{P}_1|\dots|\underline{P}_B|}$$
(μ will be the renormalised mass identified later)

- Non-planar sector is scaled away:
$$\lim_{\theta\to\infty} \sum_{g=0}^{\infty} G^{(g)}_{\underline{\dots}} \equiv G^{(0)}_{\underline{\dots}} =: G_{\underline{\dots}},$$
but punctures $B > 1$ remain!

- For $\theta \to \infty$ the oscillator potential $(2\Theta^{-1}x)^2$ disappears.

- We recover translation-invariant ϕ^4_4 on ($\theta{=}\infty$)-Moyal space, i.e. ϕ^4_4 with highly non-local interaction.

- This case was studied by [Becchi-Giusto-Imbimbo, 2003] in momentum space. They called the topology 'swiss cheese'.

Problem 1: Translate results from matrix to momentum space!

The renormalised 2-point function

SD-equation is non-linear integral equation for $G_{ab}[Z, \mu_{bare}]$ alone.

- The integrals diverge for finite Z, μ_{bare}. We repair this by normalisation conditions
  - (1) $G_{00} = 1$ adjusting the renormalised mass μ,
  - (2) $\frac{d}{da} G_{a0}\big|_{a=0} = -1$ adjusting prefactor of Laplacian.
- (1) and divergent part of (2) are processed, leaving finite part $\mathcal{Y} := \frac{\lambda}{64\pi^2} \lim_{b\to 0} \frac{1}{b} \int_0^{\Lambda^2} dp\, (G_{p0} - G_{pb})$.

The normalised integral equation is cubic in G_{ab}, but its difference to the boundary equation is quadratic:

$$(G_{ab} - G_{a0})(1 + \mathcal{Y} + bG_{a0}) + bG_{a0}^2$$
$$= \frac{\lambda}{64\pi^2} \int_0^{\Lambda^2} dp\, \frac{p(G_{pb} - G_{p0})G_{a0} - a(G_{ab} - G_{a0})G_{p0}}{p - a}$$

Assuming G_{ab} Hölder-continuous, the integral is rearranged:

$$\Big(\frac{b}{a}+\frac{1+\mathcal{Y}+\frac{\lambda a}{64\pi}\mathcal{H}_a\big[G_{\bullet 0}\big]}{aG_{a0}}\Big)D_{ab}-\frac{\lambda}{64\pi}\mathcal{H}_a\big[D_{\bullet b}\big]=-G_{a0},$$

$$-\frac{\lambda}{64\pi}\mathcal{H}_0[D_{\bullet 0}]=\mathcal{Y}$$

where $D_{ab}:=\frac{a}{b}(G_{ab}-G_{a0})$

Finite Hilbert transform $\mathcal{H}_a[f(\bullet)]:=\frac{1}{\pi}\lim_{\epsilon\to 0}\Big(\int_0^{a-\epsilon}+\int_{a+\epsilon}^{\Lambda^2}\Big)\frac{f(q)\,dq}{q-a}$

- preserves $L^p[0,\Lambda^2]$ for $p>1$, not for $p=1$ [M. Riesz, 1928], $\|\mathcal{H}\|_{L^p\to L^p}=\max(\tan\frac{\pi}{2p},\cot\frac{\pi}{2p})$ [Pichorides, 1972]
- does not preserve $\mathcal{C}[0,\Lambda^2]$
- preserves locally-Hölder* spaces $(L^p\cap H_\eta)(]0,\Lambda^2[)$ [Okada-Elliott, 1994]

$f\in H_\eta[0,\Lambda^2]\;\Leftrightarrow\;\|f\|_\eta=\sup_{0\le a\le\Lambda^2}|f(a)|+\sup_{0\le a<b\le\Lambda^2}\frac{|f(b)-f(a)|}{(b-a)^\eta}<\infty$

The Carleman equation

Theorem [Carleman 1922, Tricomi 1957]

The singular linear integral equation

$$h(x)y(x) - \hat{\lambda}\pi\mathcal{H}_x[y] = f(x)\,, \qquad x \in [-1,1]$$

is for $h(x)$ continuous + Hölder near ± 1 and $f \in L^p$ solved by

$$y(x) = \frac{\sin(\theta(x))}{\hat{\lambda}\pi}\Big(f(x)\cos(\theta(x)) + e^{\mathcal{H}_x[\theta]}\mathcal{H}_x\Big[e^{-\mathcal{H}_\bullet[\theta]}f(\bullet)\sin(\theta(\bullet))\Big] + \frac{Ce^{\mathcal{H}_x[\theta]}}{1-x}\Big)$$

$$\theta(x) = \operatorname*{arctan}_{[0,\pi]}\Big(\frac{\hat{\lambda}\pi}{h(x)}\Big)\,, \quad \sin(\theta(x)) = \frac{|\hat{\lambda}\pi|}{\sqrt{(h(x))^2 + (\hat{\lambda}\pi)^2}}$$

where C is an arbitrary constant.

Assumption: $C = 0$

The breakthrough

Theorem

$$G_{ab} = 64\pi(1+\mathcal{Y})\frac{\sin(\theta_b(a))}{|\lambda| a} e^{\mathcal{H}_a[\theta_b(\bullet)] - \mathcal{H}_0[\theta_0(\bullet)]}$$

$$\frac{\mathcal{Y}}{1+\mathcal{Y}} = \frac{\lambda}{64\pi^2}\int_0^{\Lambda^2} dp\, \frac{\sin^2(\theta_0(p))}{(\frac{\lambda p}{64\pi})^2}$$

$$\theta_b(a) := \operatorname*{arctan}_{[0,\pi]}\left(\frac{\frac{\lambda a}{64\pi}}{b + \frac{1+\mathcal{Y}+\frac{\lambda a}{64\pi}\mathcal{H}_a[G_{\bullet 0}]}{G_{a0}}}\right) \qquad (*)$$

Consequence: $G_{ab} \geq 0$!

Main steps of the proof:

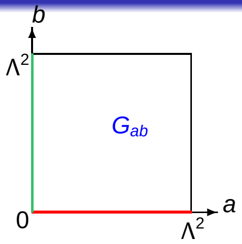
1. (*) is Carleman eq. $\frac{\lambda}{64\pi}\cot\theta_0(a) G_{a0} - \frac{\lambda}{64\pi}\mathcal{H}_a[G_{\bullet 0}] = \frac{1+\mathcal{Y}}{a}$
2. Tricomi's identity

$$e^{-\mathcal{H}_a[\theta_b]}\cos(\theta_b(a)) + \mathcal{H}_a\left[e^{-\mathcal{H}_\bullet[\theta_b]}\sin(\theta_b(\bullet)\right] = 1$$

The self-consistency equation

Given boundary value G_{a0},
Carleman computes G_{ab},
in particular G_{0b}

symmetry forces $G_{b0} = G_{0b}$



Master equation

The theory is completely determined by the solution of the fixed point equation (with $\mathcal{Y}$ determined by $\frac{dG_{b0}}{db}\Big|_{b=0} = -1$)

$$G_{b0} = \frac{1+\mathcal{Y}}{1+b+\mathcal{Y}} \exp\left(-\frac{\lambda}{64\pi^2} \int_0^b dt \int_0^\infty \frac{dp}{\left(\frac{\lambda p}{64\pi^2}\right)^2 + \left(t + \frac{1+\mathcal{Y}+\frac{\lambda p}{64\pi}\mathcal{H}_p[G_{\bullet 0}]}{G_{p0}}\right)^2} \right)$$

Problem 2 (Analysis): Rigorously prove existence and uniqueness of solution G_{b0} in Hölder space!

Correlation functions for $B = 1$ punctures

Schwinger-Dyson equation for $G_{ab_1\dots b_{N-1}}$

$$\left(\frac{b_1}{a} + \frac{1+\frac{\lambda\mathcal{Y}}{64\pi^2}+\frac{\lambda a}{64\pi}\mathcal{H}_a[G_{\bullet 0}]}{aG_{a0}}\right)\cdot(aG_{ab_1\dots b_{N-1}}) - \frac{\lambda}{64\pi}\mathcal{H}_a[\bullet G_{\bullet b_1\dots b_{N-1}}]$$
$$= \lambda \sum_{l=1}^{\frac{N-2}{2}} G_{b_1\dots b_{2l}} \frac{G_{b_{2l}b_{2l+1}\dots b_{N-1}} - G_{ab_{2l+1}\dots b_{N-1}}}{b_{2l} - a}$$

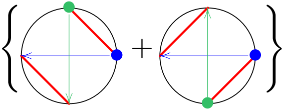
- This is again a Carleman equation, with identical linear part as for ▸ two-point function.
- Reality $\mathcal{Z} = \overline{\mathcal{Z}}$ implies invariance under orientation reversal $G_{ab_1\dots b_{N-1}} = G_{b_{N-1}\dots b_1 a} = G_{ab_{N-1}\dots b_1}$

Theorem (algebraic recursion formula for N-point function)

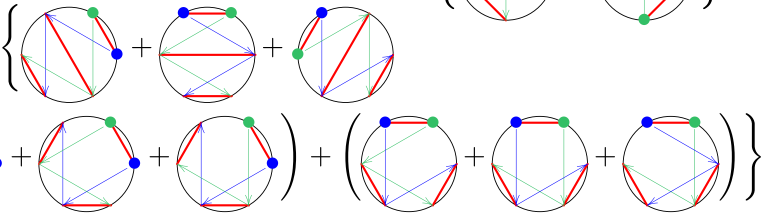
$$G_{b_0b_1\dots b_{N-1}} = (-\lambda)\sum_{l=1}^{\frac{N-2}{2}} \frac{G_{b_0b_1\dots b_{2l-1}}G_{b_{2l}b_{2l+1}\dots b_{N-1}} - G_{b_{2l}b_1\dots b_{2l-1}}G_{b_0b_{2l+1}\dots b_{N-1}}}{(b_0-b_{2l})(b_1-b_{N-1})}$$

Graphical realisation

$$G_{b_0b_1b_2b_3} = (-\lambda)\frac{G_{b_0b_1}G_{b_2b_3} - G_{b_0b_3}G_{b_2b_1}}{(b_0-b_2)(b_1-b_3)} = -\lambda\Big\{ \ldots + \ldots \Big\}$$



$$G_{b_0\dots b_5} = \lambda^2\Big\{ \ldots \Big\}$$



b_i —— b_j $= G_{b_ib_j}$ leads to non-crossing chord diagrams; these are counted by the Catalan number $C_{\frac{N}{2}} = \frac{N!}{(\frac{N}{2}+1)!\frac{N}{2}!}$

b_i ⟶ b_j $= \frac{1}{b_i - b_j}$ leads to rooted trees connecting the even or odd vertices, intersecting the chords only at vertices

Problem 3 (Combinatorics): Which trees arise for a given chord diagram?

The effective coupling constant

Proposition

The effective coupling constant $\lambda_{\mathit{eff}} = -G^0_{0000}$ of ϕ^4_4-theory on Moyal$_{\theta=\infty}$ is given in terms of the bare coupling constant λ by

$$\lambda_{\mathit{eff}} = \lambda\left(1 + \frac{\lambda}{64\pi^2}\int_0^\infty dp\, \frac{(\frac{1-G_{p0}}{p} - G_{p0})G_{p0}}{(\frac{\lambda p}{64\pi}G_{p0})^2 + (1+\mathcal{Y}+\frac{\lambda p}{64\pi}\mathcal{H}_p[G_{\bullet 0}])^2}\right)$$

- Assuming the master equation for G_{b0} to be solvable, the change $\lambda_{\mathit{eff}} \mapsto \lambda$ is only a finite renormalisation of λ_{eff} in response to an infinite change of scales.
- Consequently, the theory has a non-perturbatively vanishing β-function, although it is not exactly scale-invariant.

Functions with $B \geq 2$ punctures

- By reality, $(N_1 + \ldots + N_B)$-point functions with one $N_i > 2$ are purely algebraic, e.g.

$$G_{abc|d} = \lambda \frac{G_{a|d}G_{cb} - G_{b|d}G_{ca}}{(b-c)(b-a)} + \lambda \frac{G_{ba}G_{c|d} - G_{bc}G_{a|d}}{(b-c)(c-a)} + \lambda \frac{G_{abcd} - G_{dbca}}{(b-c)(d-a)}$$

$$G_{abcd|ef} = \lambda \frac{G_{ba}G_{cd|ef} - G_{bc}G_{ad|ef} + G_{ba|ef}G_{cd} - G_{bc|ef}G_{ad}}{(c-a)(b-d)} + \lambda \frac{G_{abcdef} - G_{ebcdaf}}{(e-a)(b-d)} + \lambda \frac{G_{eabcdf} - G_{efbcda}}{(f-a)(b-d)}$$

- They are expressed in terms of $(N_1 + \ldots + N_B)$-point functions with all $N_i \leq 2$. These base functions are solutions of new Carleman equations; their solutions are explicit functions of G_{ab}.

Problem 4: This is explicitly checked only for $B = 2$ and to be extended to $B > 2$.

More open problems

Problem 5 (Analysis): The homogeneous Carleman equation has non-trivial solutions not taken into account. They arise from a winding number and seem to be relevant for $\lambda > 64\pi$.

Problem 6 (Physics): So far this is a Euclidean quantum field theory (no time). Is there an analytic continuation to a true relativistic quantum field theory?

Problem 7 (Integrability): Is there a known integrable model which explains these results, in analogy to the KdV equation for the Kontsevich model?

Problem 8 (Algebraic geometry): What topic in algebraic geometry does the M^4-matrix model compute, in analogy to the intersection numbers for the Kontsevich model?

Summary

- We have found the exact solution of a Euclidean 4D-quantum field theory. This came completely unexpected.
- The solution is presumably of little interest for physics. Its relevance lies in the mathematical structure which is not yet understood.
- The solved model is a rich cousin of the Kontsevich model. It might be of similar importance in algebraic geometry, integrability and combinatorics.
- The expansion of the exact solution at $\lambda = 0$ agrees with the Feynman graph computation, which order by order has bad behaviour whereas the exact solution is fine.
- We see this as motivation that looking for alternatives to perturbative quantum field theory in 4D is not hopeless.