

INTERTWINING OPERATORS FOR SYMMETRIC HYPERBOLIC SYSTEMS ON GLOBALLY HYPERBOLIC MANIFOLDS

by

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Abstract

In this paper, a geometric process to compare solutions of symmetric hyperbolic systems on (possibly different) globally hyperbolic manifolds is realized via a family of intertwining operators. By fixing a suitable parameter, it is shown that the resulting intertwining operator preserves Hermitian forms naturally defined on the space of homogeneous solutions. As an application, we investigate the action of the intertwining operators in the context of algebraic quantum field theory. In particular, we provide a new geometric proof for the existence of the so-called Hadamard states on globally hyperbolic manifolds.

Keywords: Symmetric hyperbolic systems, Dirac operators, wave equations, Cauchy problem, Green operators, intertwining operators, algebraic quantum field theory.

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1 Introduction

Symmetric hyperbolic systems are an important class of first-order linear differential operators acting on sections of vector bundles on Lorentzian manifolds. The most prominent examples are the classical Dirac operator and the geometric wave operator, which can be understood by reducing a suitable second-order normally hyperbolic differential operator to a first-order differential operator. In the class of Lorentzian manifolds with empty boundary known as *globally hyperbolic*, the Cauchy problem of a symmetric hyperbolic system is well-posed. As a consequence the existence of advanced and retarded Green operators is guaranteed. These operators are of essential importance in the quantization of a classical field theory: Indeed they implement the canonical commutation relation for a bosonic field theory or the canonical anti-commutation relation for fermionic field theory. Moreover, their difference, dubbed causal propagator (or Pauli-Jordan commutator), can be used to construct quantum states. For further details, we recommend the recent reviews [4, 10, 35].

In this paper, we investigate the existence of a geometrical map connecting the space of solutions of different symmetric hyperbolic systems over (possibly different) globally hyperbolic manifolds. A summary of the main result obtained is the following (*cf.* Theorem 3.4):

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Theorem 1.1. *Let $\alpha \in \{0, 1\}$ and $M_\alpha = (M, g_\alpha)$ be globally hyperbolic manifolds admitting the same Cauchy temporal function. Consider the symmetric hyperbolic systems S_α over M_α acting on sections of a real (or complex) vector bundle E_α endowed with a non-degenerate sesquilinear fiber metric. If there exists a vector bundle isometry $\kappa : E_0 \rightarrow E_1$ and the set of timelike vectors for g_1 is contained in the one for g_0 , then the spaces of inhomogeneous solutions for S_0 and S_1 are isomorphic.*

Remark 1.2. As we shall see in more details in Theorem 3.4, actually there exists a one-parameter family of isomorphisms $R^\varrho : \text{Sol}(S_0) \rightarrow \text{Sol}(S_1)$, where $\varrho \in C^\infty(M, \mathbb{R})$ is a strictly positive smooth function which plays a fundamental role in the conservation of Hermitian forms naturally defined on the space of spacially compact solutions (*cf.* Section 3.1). Unfortunately, none of these isomorphisms is canonical, in the sense that they depends on the choice of a smooth function $\chi \in C^\infty(M, [0, 1])$.

Let us briefly comment on the geometric setting. First of all, the globally hyperbolic manifolds and the vector bundles can coincide, as for the case of scalar wave equations propagating on the same manifold which differs by a smooth potential. In this case, our analysis incorporates the results of Dappiaggi and Drago in [23]. If the manifolds do not coincides but the vector bundles do, then a relation between the causal cones for the different metrics is need it in order to defined a suitable ‘intertwining’ symmetric hyperbolic systems (*cf.* Lemma 3.3). Finally, if also the vector bundles do not coincides, as for the case of spinor bundles, then an isometry is need it in order to compare the different symmetric hyperbolic systems. Let us recall that the existence of a spinor bundle isometry in the Lorentzian setting is guaranteed by the result of Bär, Gauduchon and Moroianu [3]. As a by-product of our analysis, we expect that the *intertwining operator* defined in Proposition 3.9 can be used to probe spectral properties of the Riemannian Dirac operator as follows: Consider two Riemannian Dirac operators D_α acting on sections of spinors bundles $S\Sigma_\alpha$ over a compact Riemannian manifold (Σ, h_α) . By defining $(M_\alpha = \mathbb{R} \times \Sigma_\alpha, g_\alpha = -dt^2 + h_\alpha)$ we immediately obtain a globally hyperbolic manifold. On M_α consider the following symmetric hyperbolic system

$$S_\alpha = \partial_t - \imath D_\alpha : E_\alpha \rightarrow E_\alpha,$$

where E_α is also considered as a Hermitian vector bundle on M_α via the pull-back along the projection $\pi : M_\alpha \rightarrow \Sigma_\alpha$. Since D_α is essentially self-adjoint on $L^2(S\Sigma_\alpha)$, any vector in the kernel of S_α can be written employing the spectral calculus, namely

$$\Psi_\alpha = \exp(-\imath t D_\alpha) f_\alpha, = \left(\int_{\sigma(D_\alpha)} e^{-\imath \omega t} dE_\omega \right) f_\alpha$$

where $\sigma(D_\alpha)$ and dE_ω are respectively the spectral measure and the spectrum of D_α and f_α are initial data. It follows that intertwining operators should interplay between the spectral measure of D_0 and D_1 respectively. Very recently, Capoferri and Vassiliev gave an explicit formula for the Dirac evolution operator on any 3-dimensional oriented close Riemannian manifold in [22]. Combining their results with ours, we expect to get a better understanding of eigenvalues of the Dirac operators.

Our main result has a deep implication in free quantum field theory over generic globally hyperbolic spacetimes. Indeed, by denoting with $\mathfrak{A}_\alpha^{CCR}$ (*resp.* $\mathfrak{A}_\alpha^{CAR}$) the algebra of real scalar fields (*resp.* Dirac fields) over a globally hyperbolic spacetimes M_α , the isomorphism R_ϱ defined in Proposition 3.9 (*resp.* Proposition 3.13) can be lift to a $*$ -isomorphism $\mathfrak{A}_0^{CCR} \simeq \mathfrak{A}_1^{CCR}$ (*resp.* $\mathfrak{A}_0^{CAR} \simeq \mathfrak{A}_1^{CAR}$) – *cf.* Theorem 4.3 (*resp.* Theorem 4.6). Remarkably, the pullback of a quasifree state along this $*$ -isomorphism preserves the singular structure of the two-point distribution associated to the state (*cf.* Theorem 4.12). This result is used to provide a new geometrical proof of the existence of the so-called Hadamard states (*cf.* Corollary 4.14).

This paper is structured as follows. In Section 2 we review well know-facts about symmetric hyperbolic systems. In particular in the Subsection 2.1.1 and 2.1.2 we introduce respectively

the (classical) Dirac operator and the geometric wave operator as main examples of symmetric hyperbolic systems. In Section 3 we prove the main result of this paper and we investigate the conservation of Hermitian forms. Finally, Section 4 is devoted to analyzing the consequence of such isomorphism in the context of algebraic quantum field theory.

Notation and convention

- The symbol \mathbb{K} denotes one of the elements of the set $\{\mathbb{R}, \mathbb{C}\}$;
- $M_g := (M, g)$ is a globally hyperbolic manifold (with empty boundary) – *cf.* Definition 2.1 – and g has the signature $(-, + \dots, +)$;
- \mathcal{GH}_M denotes the space of globally hyperbolic metrics on a smooth manifold M such that, for any $g_0, g_1 \in \mathcal{GH}_M$, $M_0 = (M, g_0)$ and $M_1 = (M, g_1)$ have the same Cauchy temporal function;
- E is a \mathbb{K} -vector bundle over M_g with N -dimensional fibers, denoted by E_p for $p \in M$, and endowed with a nondegenerate sesquilinear fiber metric $\langle \cdot | \cdot \rangle: \Gamma(E_p) \times \Gamma(E_p) \rightarrow \mathbb{K}$;
- $\Gamma_c(E), \Gamma_{pc}(E), \Gamma_{fc}(E), \Gamma_{tc}(E), \Gamma_{sc}(E)$ resp. $\Gamma(E)$ denote the spaces of compactly supported, past compactly supported, future compactly supported, timelike compactly supported, spacelike compactly supported resp. smooth sections of E ;
- $S: \Gamma(E) \rightarrow \Gamma(E)$ is a symmetric hyperbolic system – *cf.* Definition 2.5 – and S^\dagger, S^* are respectively the formal adjoint and the formal dual operator of S – *cf.* Remark 2.7;
- when M is spin, SM_g denotes the spinor bundle – *cf.* Definition 2.13 – and $D: \Gamma(SM_g) \rightarrow \Gamma(SM_g)$ denotes the classical Dirac operator – *cf.* Definition 2.14;
- for a fixed, but arbitrary $f \in \Gamma(E)$, $\text{Sol}(S)$ denotes the space of inhomogeneous solutions

$$\text{Sol}(S) = \{\Psi \in \Gamma(E) \mid S\Psi = f \text{ with } f \in \Gamma(E)\},$$

while $\text{Sol}_{sc}(S)$ denotes the space of homogeneous solutions with spacially compact support

$$\text{Sol}_{sc}(S) = \{\Psi \in \Gamma_{sc}(E) \mid \Psi \in \ker S\}.$$

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2 Symmetric hyperbolic systems

The goal of this section is to present a self-contained overview of symmetric hyperbolic systems and their properties on Lorentzian manifolds. For a detailed introduction, we recommend the lecture notes of Bär [2].

On a generic Lorentzian manifold, the Cauchy problem for a differential operator is in general ill-posed: This can be a consequence of the presence of closed timelike curves or the presence of naked singularities. Therefore, it is convenient to restrict oneself to the class of *globally hyperbolic manifolds*.

Definition 2.1. A *globally hyperbolic manifold* is a $(n + 1)$ -dimensional, oriented, time-oriented, smooth Lorentzian manifold (M, g) such that

- (i) There are no closed causal curves;
- (ii) For every point $p, q \in M$, $J^+(p) \cap J^-(q)$ is compact;

where $J^+(U)$ (*resp.* $J^-(U)$) denotes the set of points of M that can be reached by future (*resp.* past) directed causal curves starting from $U \subset M$.

Notation 2.2. For the rest of this section, $M_g := (M, g)$ will always denote a globally hyperbolic manifold and we adopt the convention that the metric g has signature $(-, +, \dots, +)$.

The class of globally hyperbolic manifolds contains many important spacetimes, e.g. Minkowski spacetime, Friedmann-Robertson-Walker models, the Schwarzschild blackhole and de Sitter space. In [41], Geroch established the equivalence for a Lorentzian manifold being global hyperbolic and the existence of a *Cauchy hypersurface* Σ (i.e. an achronal subset which is crossed exactly once by any inextendible timelike curve), which implies that M is homeomorphic to $\mathbb{R} \times \Sigma$ and all Cauchy hypersurfaces are homeomorphic. The proof was carried out by finding a Cauchy time function, namely a continuous function $t : M \rightarrow \mathbb{R}$ which increases strictly on any future-directed causal curve such that each level $t^{-1}(t_0)$, $t_0 \in \mathbb{R}$, is a Cauchy hypersurface. In [17] Bernal and Sánchez “smoothened” the result of Geroch by introducing the notion of *Cauchy temporal function*.

Definition 2.3. We say that a smooth time function $t : M \rightarrow \mathbb{R}$ is a *Cauchy temporal function* if its gradient ∇t is past-directed timelike and its level set is a smooth Cauchy hypersurface.

Theorem 2.4 ([17], Theorem 1.1 and Theorem 1.2). *Any globally hyperbolic manifold admits a Cauchy temporal function. In particular, it is isometric to the smooth product manifold $\mathbb{R} \times \Sigma$ with metric*

$$g = -\beta^2 dt^2 + h_t$$

where $t : \mathbb{R} \times \Sigma \rightarrow \mathbb{R}$ is Cauchy temporal function, $\beta : \mathbb{R} \times \Sigma \rightarrow \mathbb{R}$ is a smooth positive function and h_t is a Riemannian metric on each level set of t .

Let now E be a \mathbb{K} -vector bundle over a globally hyperbolic manifold M_g with finite rank N and endowed with a (possibly indefinite) non-degenerate sesquilinear fiber metric

$$\langle \cdot | \cdot \rangle : E_p \times E_p \rightarrow \mathbb{K}. \quad (2.1)$$

Definition 2.5 ([1], Definition 5.1). A linear differential operator $S : \Gamma(E) \rightarrow \Gamma(E)$ of first order is called a *symmetric hyperbolic system* over M_g if

- (S) The principal symbol $\sigma_S(\xi) : E_p \rightarrow E_p$ is Hermitian with respect to $\langle \cdot | \cdot \rangle$ for every $\xi \in T_p^*M$ and for every $p \in M$;
- (H) For every future-directed timelike covector $\tau \in T_p^*M$, the bilinear form $\langle \sigma_S(\tau) \cdot | \cdot \rangle$ is positive definite on E_p .

Remark 2.6. Notice that Definition 2.5 depends on the fiber metric $\langle \cdot | \cdot \rangle$ and on the Lorentzian metric g which defined the set of future-directed timelike covectors.

Let us recall that for a first-order linear differential operator $S : \Gamma(E) \rightarrow \Gamma(E)$ the principal symbol $\sigma_S : T^*M \rightarrow \text{End}(E)$ can be characterized by

$$S(fu) = fSu + \sigma_S(df)u \quad (2.2)$$

where $u \in \Gamma(E)$ and $f \in C^\infty(M)$. If we choose local coordinates (t, x^1, \dots, x^n) on M , with x^i local coordinates on Σ_t , and a local trivialization of E , any linear differential operator $S : \Gamma(E) \rightarrow \Gamma(E)$ of first order reads in a point $p \in M$ as

$$S := A_0(p)\partial_t + \sum_{j=1}^n A_j(p)\partial_{x^j} + B(p) \quad (2.3)$$

where the coefficients A_0, A_j, B are $N \times N$ matrices, with N being the rank of E , depending smoothly on $p \in M$. In these coordinates, Condition (S) in Definition 2.5 reduces to

$$\langle A_0 \cdot | \cdot \rangle = \langle \cdot | A_0 \cdot \rangle \quad \text{and} \quad \langle A_j \cdot | \cdot \rangle = \langle \cdot | A_j \cdot \rangle$$

for $j = 1, \dots, n$. Condition (H) can be stated as follows: For any future directed, timelike covector $\tau = dt + \sum_j \alpha_j dx^j$,

$$\langle \sigma_S(\tau) \cdot | \cdot \rangle = \langle (A_0 + \sum_{j=1}^n \alpha_j A_j) \cdot | \cdot \rangle$$

defines a scalar product on E_p .

Remark 2.7. Let S^\dagger be the *formal adjoint* operator of S , i.e. the unique linear differential operator $S^\dagger : \Gamma(E) \rightarrow \Gamma(E)$ defined by

$$\int_M \langle \Psi | S\Phi \rangle \text{vol}_M = \int_M \langle S^\dagger \Psi | \Phi \rangle \text{vol}_M$$

for every $\Phi, \Psi \in \Gamma(E)$ with $\text{supp } \Phi \cap \text{supp } \Psi$ compact. Its principal symbol σ_{S^\dagger} satisfies

$$\sigma_{S^\dagger}(\xi) = -\sigma_S(\xi)^t,$$

for every $\xi \in T^*M$, see e.g. [2, Lemma 1.1.26]. By using property (S) in Definition 2.5 it follows that $-S^\dagger$ is a symmetric hyperbolic system. Similarly, let S^* be the *formal dual* operator of S , i.e. the unique linear differential operator acting on sections of the dual vector bundle $S^* : \Gamma(E^*) \rightarrow \Gamma(E^*)$ defined by

$$\int_M \Phi^*(S\Psi) \text{vol}_M = \int_M (S^*\Phi^*)\Psi \text{vol}_M$$

for every $\Phi^* \in \Gamma(E^*)$, $\Psi \in \Gamma(E)$ with $\text{supp } \Phi^* \cap \text{supp } \Psi$ compact. By introducing the vector bundle isometry

$$\Upsilon : E \rightarrow E^* \quad \Phi \mapsto \Phi^* := \langle \Phi | \cdot \rangle,$$

computations show that $S^* = \Upsilon S^\dagger \Upsilon^{-1}$. For further details we refer to [2, Section 1.1.3].

Theorem 2.8 ([1], Theorem 5.6 and Proposition 5.7). *The Cauchy problem for a symmetric hyperbolic system S on a globally hyperbolic manifold is well-posed, i.e. for any $f \in \Gamma_c(E)$ and $h \in \Gamma_c(E|_{\Sigma_0})$ there exists a unique smooth solution $\Psi \in \Gamma_{sc}(E)$ with spatially compact support to the initial value problem*

$$\begin{cases} S\Psi = f \\ \Psi|_{\Sigma_0} = h \end{cases} \quad (2.4)$$

which depends continuously on the data (f, h) .

As a byproduct of the well-posedness of the Cauchy problem, it follows the existence of Green operators.

Proposition 2.9 ([1], Theorem 5.9 and Theorem 3.8). *A symmetric hyperbolic system is Green hyperbolic, i.e. there exist linear maps, dubbed advanced Green operator $G^+ : \Gamma_{pc}(E) \rightarrow \Gamma_{pc}(E)$ and retarded Green operator $G^- : \Gamma_{fc}(E) \rightarrow \Gamma_{fc}(E)$, satisfying*

- (i.a) $G^+ \circ S f = S \circ G^+ f = f$ for all $f \in \Gamma_{pc}(E)$,
- (ii.a) $\text{supp}(G^+ f) \subset J^+(\text{supp } f)$ for all $f \in \Gamma_{pc}(E)$;
- (i.b) $G^- \circ S f = S \circ G^- f = f$ for all $f \in \Gamma_{fc}(E)$,
- (ii.b) $\text{supp}(G^- f) \subset J^-(\text{supp } f)$ for all $f \in \Gamma_{fc}(E)$.

Remark 2.10. On account of Remark 2.7 it immediately follows that the formal dual operator S^* is Green hyperbolic

$$\begin{aligned} G^{+*} &: \Gamma_{pc}(\mathbf{E}^*) \cap \Gamma_{sc}(\mathbf{E}^*) \rightarrow \Gamma_{pc}(\mathbf{E}^*) \cap \Gamma_{sc}(\mathbf{E}^*) \\ G^{-*} &: \Gamma_{fc}(\mathbf{E}^*) \cap \Gamma_{sc}(\mathbf{E}^*) \rightarrow \Gamma_{pc}(\mathbf{E}^*) \cap \Gamma_{sc}(\mathbf{E}^*) \end{aligned}$$

A straightforward computation shows that

$$\int_M (G^{+*}\Phi_1^*)(\Psi_1)\text{vol}_M = \int_M \Phi_1^*(G^-\Psi_1)\text{vol}_M \quad \int_M (G^{-*}\Phi_2^*)(\Psi_2)\text{vol}_M = \int_M \Phi_2^*(G^+\Psi_2)\text{vol}_M,$$

for every $\Phi_1^* \in \text{dom}(G^{+*})$, $\Phi_2^* \in \text{dom}(G^{-*})$, $\Psi_1 \in \text{dom}(G^-)$ and $\Psi_2 \in \text{dom}(G^+)$.

Definition 2.11. Let S be a Green hyperbolic operator and denote with G^+ and G^- respectively the advanced and retarded Green operator. We call *causal propagator* $G : \Gamma_{fc}(\mathbf{E}) \rightarrow \Gamma(\mathbf{E})$ the operator defined by $G := G^+ - G^-$.

The causal propagator characterize the space of solutions to the homogeneous Cauchy problem, namely for any $f \in \Gamma_{tc}(\mathbf{E})$, $\Psi := Gf \in \ker S$. The properties of the causal propagator are summarized in the following proposition.

Proposition 2.12 ([4], Theorem 3.5). *Let G be the causal propagator for a Green hyperbolic operator $S : \Gamma(\mathbf{E}) \rightarrow \Gamma(\mathbf{E})$. Then the following linear maps forms an exact sequence*

$$\{0\} \rightarrow \Gamma_{tc}(\mathbf{E}) \xrightarrow{S} \Gamma_{tc}(\mathbf{E}) \xrightarrow{G} \Gamma(\mathbf{E}) \xrightarrow{S} \Gamma(\mathbf{E}) \rightarrow \{0\}.$$

2.1 Geometric examples

In this section, we shall review two of the most important examples of symmetric hyperbolic systems: the classical Dirac operator and the geometric wave operator. More examples can be found in [42, 47], while for further details on spin geometry on Lorentzian spin manifold, we refer to [3, 25, 49].

2.1.1 The classical Dirac operator

Let M_g be a globally hyperbolic manifold and assume to have a spin structure *i.e.* a twofold covering map from the $\text{Spin}_0(1, n)$ -principal bundle P_{Spin_0} to the bundle of positively-oriented tangent frames P_{SO^+} of M such that the following diagram is commutative:

$$\begin{array}{ccc} P_{\text{Spin}_0} \times \text{Spin}_0(1, n) & \longrightarrow & P_{\text{Spin}} \\ \downarrow & & \downarrow \\ P_{\text{SO}^+} \times \text{SO}(1, n) & \longrightarrow & P_{\text{SO}^+} \longrightarrow M_g. \end{array}$$

The existence of spin structures is related to the topology of M_g . A sufficient (but not necessary) condition for the existence of a spin structure is the parallelizability of the manifold. Therefore, since any 3-dimensional orientable manifold is parallelizable, it follows by Theorem 2.4 that any 4-dimensional globally hyperbolic manifold admits a spin structure. Given a fixed spin structure, one can use the spinor representation to construct the spinor bundle

Definition 2.13. Let M_g be a (globally hyperbolic) spin manifold. The (*complex*) *spinor bundle* is the complex vector bundle

$$\text{SM}_g := \text{Spin}_0(1, n) \times_{\rho} \mathbb{C}^N$$

where $\rho : \text{Spin}_0(1, n) \rightarrow \text{Aut}(\mathbb{C}^N)$ is the complex $\text{Spin}_0(1, n)$ representation and $N := 2^{\lfloor \frac{n+1}{2} \rfloor}$.

The spinor bundle is enriched with the following structure:

- a natural $\text{Spin}_0(1, n)$ -invariant indefinite fiber metrics

$$\langle \cdot | \cdot \rangle: \mathbb{S}_p \mathbb{M}_g \times \mathbb{S}_p \mathbb{M}_g \rightarrow \mathbb{C};$$

- a Clifford multiplication, i.e. a Clifford multiplication, i.e. a fiber-preserving map

$$\gamma: \text{TM} \rightarrow \text{End}(\mathbb{S}\mathbb{M}_g)$$

which satisfies for all $p \in \mathbb{M}_g$, $u, v \in \text{T}_p \mathbb{M}$ and $\psi, \phi \in \mathbb{S}_p \mathbb{M}_g$

$$\gamma(u)\gamma(v) + \gamma(v)\gamma(u) = -2g(u, v)\text{Id}_{\mathbb{S}_p \mathbb{M}_g} \quad \text{and} \quad \langle \gamma(u)\psi | \phi \rangle = \langle \psi | \gamma(u)\phi \rangle. \quad (2.5)$$

Using the spin product (2.1.1), we denote as *adjunction map*, the complex anti-linear vector bundle isomorphism by

$$\Upsilon_p: \mathbb{S}_p \mathbb{M}_g \rightarrow \mathbb{S}_p^* \mathbb{M}_g \quad \psi \mapsto \langle \psi | \cdot \rangle, \quad (2.6)$$

where $\mathbb{S}_p^* \mathbb{M}_g$ is the so-called *cospinor bundle*, i.e. the dual bundle of $\mathbb{S}_p \mathbb{M}_g$.

Definition 2.14. The (classical) Dirac operator D is the operator defined as the composition of the metric connection $\nabla^{\mathbb{S}\mathbb{M}_g}$ on $\mathbb{S}\mathbb{M}_g$, obtained as a lift of the Levi-Civita connection on TM , and the Clifford multiplication:

$$D = \gamma \circ \nabla^{\mathbb{S}\mathbb{M}_g}: \Gamma(\mathbb{S}\mathbb{M}_g) \rightarrow \Gamma(\mathbb{S}\mathbb{M}_g).$$

In local coordinates and with a trivialization of the spinor bundle $\mathbb{S}\mathbb{M}_g$, the Dirac operator reads as

$$D\psi = \sum_{\mu=0}^n \varepsilon_\mu \gamma(e_\mu) \nabla_{e_\mu}^{\mathbb{S}\mathbb{M}_g} \psi$$

where $\{e_\mu\}$ is a local Lorentzian-orthonormal frame of TM and $\varepsilon_\mu = g(e_\mu, e_\mu) = \pm 1$.

Remark 2.15. Note that unlike differential forms, the definition of spinors (and cospinors) requires the choice of a spin structure and it depends on the metric of the underlying manifold.

Proposition 2.16. The classical Dirac operator D on globally hyperbolic spin manifolds \mathbb{M}_g is a symmetric hyperbolic system.

Proof. The principal symbol σ_D of the Dirac operator reads as

$$\sigma_D(\xi)\psi = \gamma(\xi^\sharp)\psi \quad (2.7)$$

where $\xi \in \Gamma(\text{T}^*\mathbb{M})$, $\psi \in \Gamma(\mathbb{S}\mathbb{M}_g)$ and $\sharp: \text{T}^*\mathbb{M} \rightarrow \text{TM}$ is the musical isomorphism implemented by the Lorentzian metric. Therefore, Property (S) of Definition 2.5 is verified on account of (2.5), while Property (H) follows by [30, Proposition 1.1], provided that the spin product (2.1.1) was chosen with the appropriate sign. \square

Remark 2.17. Noticed that in the literature it is often used the canonical positive-definite scalar product $\langle \cdot | \cdot \rangle_{\mathbb{C}^N}$ despite the indefinite, non-degenerate spin product (2.1.1). As a consequence, D is no longer a symmetric hyperbolic system, but $\gamma(\frac{\partial}{\partial t})D$ does satisfies the required properties, see e.g. [43, 44].

Remark 2.18. By Theorem 2.8, it follows that the space of solutions $\text{Sol}(D)$ of the Dirac equation is not trivial. Moreover, since the Dirac operator is formally skew-adjoint, i.e. $D = -D^\dagger$, by Remark 2.7 it follows that for any solution ψ of the homogeneous Dirac equation, the adjunction map Υ realizes an isomorphism of vector spaces, namely

$$\ker D \ni \psi \mapsto \Upsilon\psi \in \ker D^* \quad \text{and} \quad \ker D^* \ni \phi^* \mapsto \Upsilon^{-1}\phi^* \in \text{Sol}(D). \quad (2.8)$$

2.1.2 The geometric wave operator

Let \mathbf{V} be an Hermitian vector bundle of finite rank and consider a normally hyperbolic operator $\mathbf{P} : \Gamma(\mathbf{V}) \rightarrow \Gamma(\mathbf{V})$, *i.e.* a 2^{nd} -order linear differential operator with principal symbol $\sigma_{\mathbf{P}}$ defined by

$$\sigma_{\mathbf{P}}(\xi) = -g(\xi, \xi) \cdot \text{Id}_{\mathbf{V}},$$

for every $\xi \in \mathbb{T}^*\mathbf{M}$. Following [2, Remark 3.7.11], we shall reduce \mathbf{P} to a symmetric hyperbolic system, but first we assume, without loss of generality, that $\mathbf{M}_g = (\mathbf{M}, g)$ is given by

$$\mathbf{M} := \mathbb{R} \times \Sigma \quad g = -\beta^2 dt^2 + h_t$$

see Theorem 2.4. By [6, Lemma 1.5.5], there exists a unique (metric) connection $\nabla^{\mathbf{V}}$ on \mathbf{V} and a unique endomorphism field $c \in \Gamma(\text{End}(\mathbf{V}))$ such that

$$\mathbf{P} = \text{tr}_g(\nabla^{\mathbf{V}}\nabla^{\mathbf{V}}) + c = \frac{1}{\beta^2}\nabla_{\partial_t}^2 + b_0\nabla_{\partial_t} + (\nabla^{\Sigma})^*\nabla^{\Sigma} + \nabla_b^{\Sigma} + c, \quad (2.9)$$

where ∇^{Σ} is defined by $\nabla_X^{\Sigma} := \nabla_X^{\mathbf{V}}$ for all $X \in \mathbb{T}\Sigma$, while $b_0 \in C^{\infty}(\mathbf{M})$ and $b \in \Gamma(\mathbb{T}\Sigma)$ are given by

$$b_0 := \frac{1}{2\beta^2} \left(\text{tr}_{h_t}(\partial_t h_t) - \frac{\partial_t \beta^2}{\beta^2} \right) \quad \text{and} \quad b := -\frac{1}{2\beta^2} \text{grad}_{h_t}(\beta^2).$$

Equation (2.9) allows us to rewrite the Cauchy problem for $\mathbf{P} : \Gamma(\mathbf{V}) \rightarrow \Gamma(\mathbf{V})$

$$\begin{cases} \mathbf{P}u = f \\ u|_{\Sigma_0} = h \\ \nabla_{\partial_t} u|_{\Sigma_0} = h' \end{cases} \quad (2.10)$$

as a Cauchy problem for $\mathbf{S} : \Gamma(\mathbf{E}) \rightarrow \Gamma(\mathbf{E})$,

$$\begin{cases} \mathbf{S}\Psi := (A_0\nabla_{\partial_t}^{\mathbf{V}} + A_{\Sigma}\nabla^{\Sigma} + B)\Psi = \mathfrak{f} \\ \Psi|_{\Sigma_0} = \mathfrak{h} \end{cases} \quad (2.11)$$

where \mathbf{E} is the Hermitian vector bundle $\mathbf{E} := \mathbf{V} \oplus (\mathbb{T}^*\Sigma \otimes \mathbf{V}) \oplus \mathbf{V}$, $B \in \Gamma(\text{End}(\mathbf{E}))$ and

$$\Psi := \begin{pmatrix} \nabla_{\partial_t}^{\mathbf{V}} u \\ \nabla^{\Sigma} u \\ u \end{pmatrix}, \quad \mathfrak{f} := \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix}, \quad A_0 := \begin{pmatrix} \frac{1}{\beta^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_{\Sigma} = \begin{pmatrix} 0 & -\text{tr}_{h_t} & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$C := \begin{pmatrix} b_0 & b_{\mathbf{j}} & c \\ 0 & \frac{1}{2}h_t^{-1}\partial_t h_{t,\mathbf{j}} & R_{\partial_t, \cdot} \\ -1 & 0 & 0 \end{pmatrix}.$$

The Cauchy problem (2.11) should be read as follows: $\nabla_{\partial_t}\nabla^{\Sigma}u$ is defined by

$$(\nabla_{\partial_t}\nabla^{\Sigma}u)_X := \nabla_{\partial_t}\nabla_X^{\Sigma}u - \nabla_{(\nabla_{\partial_t}X)^{\Sigma}}u$$

for all $X \in \Gamma(\pi_2^*\mathbb{T}\Sigma)$. The term $\nabla^{\Sigma}\Psi$ is a section of $(\mathbb{T}^*\Sigma \otimes \mathbf{V}) \oplus (\mathbb{T}^*\Sigma \otimes \mathbb{T}^*\Sigma \otimes \mathbf{V}) \oplus (\mathbb{T}^*\Sigma \otimes \mathbf{V}) \rightarrow \mathbf{M}$, the trace coefficient contracting $\mathbb{T}^*\Sigma \otimes \mathbb{T}^*\Sigma$ of course. The coefficient $\frac{1}{2}h_t^{-1}\partial_t h_{t,\mathbf{j}}$ is more or less the Weingarten map put into the $\mathbb{T}\Sigma$ slot. The curvature tensor R is that of ∇ and is by convention given for all $X, Y \in \mathbb{T}\mathbf{M}$ by $R_{X,Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. As in [2, Remark 3.7.11], Conditions (S) and (H) can be easily checked. Hence \mathbf{S} is a symmetric hyperbolic system.

Remark 2.19. Notice that, while any solution u of the Cauchy problem (2.10) gives a solution Ψ to the Cauchy problem (2.11), the contrary does not hold. Indeed, the space of initial data for Ψ is “too large” and some a suitable restriction has to be imposed. For further details we refer to [2, Remark 3.7.11].

3 Intertwining operators

This section aims to generalize the results of Dappiaggi and Drago in [23] by constructing a geometric map between the solutions space of symmetric hyperbolic systems defined on (possibly different) vector bundles over (possibly different) globally hyperbolic manifolds. Since the construction of a vector bundle can depend in general on the metric of the underlying Lorentzian manifold, as for the case of classical Dirac operator, it became necessary first to find a path connecting different metrics. Despite the space of Lorentzian metrics on a fixed smooth manifold is not path-connected, when we restrict our attention to globally hyperbolic manifolds, we get the following result.

Lemma 3.1. *Let \mathcal{GH}_M be the space of globally hyperbolic metrics on a smooth manifold M such that, for any $g_0, g_1 \in \mathcal{GH}_M$, $M_0 = (M, g_0)$ and $M_1 = (M, g_1)$ have the same Cauchy temporal function. Then \mathcal{GH}_M is convex.*

Proof. Let $g_0, g_1 \in \mathcal{GH}_M$. Since M_0 and M_1 admit the same Cauchy temporal function, there exists a isometric splitting $M = \mathbb{R} \times \Sigma$ with metric $g_\alpha = -\beta_\alpha^2 dt^2 + h_{t,\alpha}$, for $\alpha \in \{0, 1\}$. This in particular, shows that the convex linear combination $g_\lambda := \lambda g_1 + (1 - \lambda) g_0$ for any $\lambda \in [0, 1]$, is a globally hyperbolic metric. \square

Keeping in mind Lemma 3.1 we introduce the following setup, which we shall use through this section:

Setup 3.2. For $\alpha \in \{0, 1\}$, we have the following:

- \mathcal{GH}_M denotes the space of globally hyperbolic metrics on a smooth manifold M such that, for any $g_0, g_1 \in \mathcal{GH}_M$, $M_0 = (M, g_0)$ and $M_1 = (M, g_1)$ have the same Cauchy temporal function;
- $M_\alpha := (M, g_\alpha)$, where $g_\alpha \in \mathcal{GH}_M$ and $g_1 \leq g_0$ (i.e. the set of timelike vectors for g_1 is contained in the one for g_0);
- E_α is a \mathbb{K} -vector bundle over M_α with finite rank and endowed with a nondegenerate sesquilinear fiber metric $\langle \cdot | \cdot \rangle_\alpha$;
- $\kappa_{1,0} : E_0 \rightarrow E_1$ is a fiberwise linear isometry of vector bundles and, for any $\varrho \in C^\infty(M, \mathbb{R})$ strictly positive, we set $\kappa_{1,0}^\varrho := \varrho \kappa_{1,0}$ and $\kappa_{0,1}^\varrho := \varrho^{-1} \kappa_{0,1}$;
- $\text{Sol}(S_\alpha)$ denotes the space of solutions for the symmetric hyperbolic system S_α over M_α

$$\text{Sol}(S_\alpha) := \{ \Psi_\alpha \in \Gamma(E_\alpha) \mid S_\alpha \Psi_\alpha = f_\alpha \text{ with } f_\alpha \in \Gamma(E_\alpha) \}.$$

To construct an intertwining operator, we need a preliminary lemma.

Lemma 3.3. *Assume the Setup 3.2. For any $\chi \in C^\infty(M, [0, 1])$, the operator defined by*

$$S_{\chi,1}^\varrho := (1 - \chi) S_{0,1}^\varrho + \chi S_1 : \Gamma(E_1) \rightarrow \Gamma(E_1) \tag{3.1}$$

is a symmetric hyperbolic system over M_1 .

Proof. Using the characterization of the principal symbols given as in Equation (2.2), for every $\Psi_1 \in \Gamma(E_1)$ and $f \in C^\infty(M)$ we thus obtain

$$\begin{aligned} \sigma_{S_{\chi,1}^\varrho}(df)\Psi_1 &= f \left((1 - \chi) S_{0,1}^\varrho + \chi S_1 \right) \Psi_1 - \left((1 - \chi) S_{0,1}^\varrho + \chi S_1 \right) (f\Psi_1) \\ &= (1 - \chi) \sigma_{S_{0,1}^\varrho}(df)\Psi_1 + \chi \sigma_{S_1}(df)\Psi_1. \end{aligned}$$

We first show that $S_{0,1}^e$ is a symmetric hyperbolic system. By Equation (2.2), for every $\Psi_\lambda \in \Gamma(E_\lambda)$ and $f \in C^\infty(M)$ we thus obtain

$$\sigma_{S_{0,1}^e}(df)\Psi_1 = fS_{0,1}^e\Psi_1 - S_{0,1}^e(f\Psi_1) = \kappa_{1,0}^e fS_0\Psi_0 - \kappa_{1,0}^e S_0(f\Psi_0) = \kappa_{1,0}^e \sigma_{S_0}(df)\Psi_0, \quad (3.2)$$

where $\Psi_0 = \kappa_{0,1}^e\Psi_1$ and $\kappa_{0,1}^e\kappa_{1,0}^e = \text{Id}$. Since $\kappa_{1,0}$ is a fiberwise linear isometry by assumption, it follows

$$\langle \sigma_{S_{0,1}^e}(\xi)\Psi_1 | \Psi_1 \rangle_1 = \langle \kappa_{1,0}^e \sigma_{S_0}(\xi)\Psi_0 | \kappa_{1,0}^e \Psi_0 \rangle_1 = \varrho^2 \langle \sigma_{S_0}(\xi)\Psi_0 | \Psi_0 \rangle_0, \quad (3.3)$$

where $\langle \cdot | \cdot \rangle_0$ is a fiberwise pairing on E_0 . Using that S_0 is a symmetric hyperbolic system, it follows immediately that $S_{0,1}^e$ satisfies property (S) in Definition 2.5. Furthermore, since $g_1 \leq g_0$, any timelike covector ξ for g_1 is also a timelike covector for g_0 . Therefore $S_{0,1}^e$ satisfies also property (H) as well. Hence it is a symmetric hyperbolic system. To conclude our proof, it is enough to notice that a convex linear combination of Hermitian operators is a Hermitian operator, and a convex linear combination of positive operators is a positive operator. Hence also $S_{\chi,1}^e$ is a symmetric hyperbolic system. \square

Building on Lemma 3.3, we now prove the main result of this paper.

Theorem 3.4. *Assume the Setup 3.2. Consider two Cauchy hypersurfaces $\Sigma^\pm \subset M_1$ such that $\Sigma_+ \subset J^+(\Sigma_-)$ and let $\chi \in C^\infty(M_1, [0, 1])$ be non-decreasing along any future-oriented timelike curve such that*

$$\chi|_{J^+(\Sigma_+)} = 1, \quad \text{and} \quad \chi|_{J^-(\Sigma_-)} = 0.$$

Finally consider the operators $R_\pm : \Gamma(E_1) \rightarrow \Gamma(E_1)$ defined by

$$R_- := \text{Id} - G_1^-(1 - \chi)(S_1 - S_{0,1}^e) \quad \text{and} \quad R_+ := \text{Id} - G_\chi^+ \chi(S_1 - S_{0,1}^e)$$

where G_χ^+ is the advanced Green operator for the operator $S_{\chi,1}^e$ defined by Equation (3.1) and G_1^- is the retarded Green operator for S_1 . If $f_1 = \kappa_{1,0}^e f_0$, then the intertwining operator

$$R_\varrho := R_- \circ R_+ \circ \kappa_{1,0}^e : \text{Sol}(S_0) \rightarrow \text{Sol}(S_1)$$

implements a non-canonical isomorphism.

Proof. Since, for any $f_1 \in \Gamma(E_1)$, $\chi f_1 \in \Gamma_{pc}(E_1)$ and $(1 - \chi)f_1 \in \Gamma_{fc}(E_1)$ then the intertwining operator R_ϱ is well defined. Indeed, for any $\Psi_0 \in \text{Sol}(S_0)$, it turns out that

$$\begin{aligned} \chi(S_1 - S_{0,1}^e) \circ \kappa_{1,0}^e \Psi_0 &\in \Gamma_{pc}(E_1) = \text{dom } G_\chi^+ \\ (1 - \chi)(S_1 - S_{0,1}^e) \circ R_+ \circ \kappa_{1,0}^e \Psi_0 &\in \Gamma_{fc}(E_1) = \text{dom } G_1^- \end{aligned}$$

The smoothness of $R_\varrho\Psi_0$ is a by-product of the regularity properties of the advanced and retarded Green operators. By straightforward computation, we thus obtain

$$S_1 \circ R_- = S_1 - (1 - \chi)(S_1 - S_{0,1}^e) = S_{\chi,1}^e \quad (3.4)$$

$$S_{\chi,1}^e \circ R_+ = S_{\chi,1}^e - \chi(S_1 - S_{0,1}^e) = S_{0,1}^e, \quad (3.5)$$

where we used that $S_1 \circ G_1^- = \text{Id}$ and $S_{\chi,1}^e \circ G_\chi^+ = \text{Id}$ (cf. Proposition 2.9). In particular, this implies that

$$S_1 \circ R_\varrho = \kappa_{1,0}^e S_0. \quad (3.6)$$

Therefore, for any $\Psi_0 \in \text{Sol}(S_0)$, it holds

$$(S_1 \circ R_\varrho)\Psi_0 = \kappa_{1,0}^e S_0\Psi_0 = \kappa_{1,0}^e f_0 = f_1.$$

Hence $R_\rho \Psi_0 \in \text{Sol}(S_1)$.

To conclude our proof, it suffices to prove that R_- and R_+ are invertible. Indeed, by defining

$$R_\rho^{-1} := \kappa_{0,1}^\rho \circ R_+^{-1} \circ R_-^{-1}$$

it follows $R_\rho \circ R_\rho^{-1} = \text{Id} : \text{Sol}(S_1) \rightarrow \text{Sol}(S_1)$ and $R_\rho^{-1} \circ R_\rho = \text{Id} : \text{Sol}(S_0) \rightarrow \text{Sol}(S_0)$. To this end, we make the following ansatz:

$$\begin{aligned} R_-^{-1} &:= \text{Id} + G_\chi^-(1 - \chi)(S_1 - S_{0,1}^\rho) : \Gamma(E_1) \rightarrow \Gamma(E_1) \\ R_+^{-1} &:= \text{Id} + G_{0,1}^+ \chi(S_1 - S_{0,1}^\rho) : \Gamma(E_1) \rightarrow \Gamma(E_1), \end{aligned}$$

where $G_{0,1}^+$ is the advanced Green operator for $S_{0,1}^\rho$. We begin by showing that R_-^{-1} is a right inverse for R_-

$$\begin{aligned} R_- \circ R_-^{-1} &= \left(\text{Id} - G_1^-(1 - \chi)(S_1 - S_{0,1}^\rho) \right) \circ \left(\text{Id} + G_\chi^-(1 - \chi)(S_1 - S_{0,1}^\rho) \right) = \\ &= \text{Id} - G_1^-(S_1 - S_{\chi,1}^\rho) + G_\chi^-(S_1 - S_{\chi,1}^\rho) - G_1^-(S_1 - S_{\chi,1}^\rho) G_\chi^-(S_1 - S_{\chi,1}^\rho) = \text{Id} \end{aligned}$$

where we used $(1 - \chi)(S_1 - S_{0,1}^\rho) = S_1 - S_{\chi,1}^\rho$ together with $G_1^-(S_1 - S_{\chi,1}^\rho) G_\chi^- = G_1^+ - G_\chi^-$. Similarly we get

$$\begin{aligned} R_+ \circ R_+^{-1} &= \left(\text{Id} - G_\chi^+ \chi(S_1 - S_{0,1}^\rho) \right) \circ \left(\text{Id} + G_{0,1}^+ \chi(S_1 - S_{0,1}^\rho) \right) = \\ &= \text{Id} - G_\chi^+(S_{\chi,1}^\rho - S_{0,1}^\rho) + G_{0,1}^+(S_{\chi,1}^\rho - S_{0,1}^\rho) - G_\chi^+(S_{\chi,1}^\rho - S_{0,1}^\rho) G_{0,1}^+(S_{\chi,1}^\rho - S_{0,1}^\rho) = \text{Id} \end{aligned}$$

where we used $\chi(S_1 - S_{0,1}^\rho) = S_{\chi,1}^\rho - S_{0,1}^\rho$ together with $G_\chi^+(S_{\chi,1}^\rho - S_{0,1}^\rho) G_{0,1}^+ = G_{0,1}^+ - G_\chi^+$. Since $R_-^{-1} \circ R_-$ and $R_+^{-1} \circ R_+$ have analogous computations, we can conclude. \square

Remark 3.5. By uniqueness of solution, R_ρ implements the following geometric map: Let be $\Psi_0 \in \text{Sol}(S_0)$ and denote with $\Psi_{1,0} = \kappa_{1,0}^\rho \Psi_0$. Then consider the operator which maps the Cauchy data $\Psi_{1,0}|_{\Sigma_-}$ to the corresponding Cauchy data on Σ_+ by evolving it via the evolution operator of $S_{\chi,1}^\rho$, which exists on account of Lemma 3.3 and Theorem 2.8. Finally define $R_\rho \Psi_0$ as the solution for $S_1 R_\rho \Psi_0 = f_1$ and Cauchy data provided by those previously obtained on Σ_+ . If the inhomogeneity $f_1 = 0$, then the evolution operator is an unitary operator from $L^2(\text{SM}|_{\Sigma_\alpha}) \rightarrow L^2(\text{SM}|_{\Sigma_\beta})$, where $\Sigma_\alpha, \Sigma_\beta \subset M$ are Cauchy hypersurfaces.

On account of Remark 2.7, the formally adjoint operator $S^\dagger : \Gamma(E) \rightarrow \Gamma(E)$ is a symmetric hyperbolic system, clearly up to a sign. Therefore the results of Theorem 3.4 applies immediately to $-S^\dagger$ and we denote its intertwining operator by R_ρ^\dagger . By denoting $\Upsilon : E \rightarrow E^*$ we immediately get the following result.

Proposition 3.6. *Assume the setup of Theorem 3.4 and let $\kappa_{1,0}^{\rho*} : \Gamma(E_0^*) \rightarrow \Gamma(E_1^*)$ be the linear isometry defined by $\kappa_{1,0}^{\rho*} := \Upsilon_1 \kappa_{1,0}^\rho \Upsilon_0^{-1}$. The intertwining operator $R_\rho^* := \Upsilon_1 R_\rho^\dagger \Upsilon_0^{-1}$ implements a non-canonical isomorphism between $\text{Sol}(S_0^*)$ and $\text{Sol}(S_1^*)$.*

Proof. As shown in Remark 2.7 it holds $S^* = \Upsilon S^\dagger \Upsilon^{-1}$, which should be reads as

$$\int_M \langle S_\alpha^\dagger \Phi_\alpha | \Psi_\alpha \rangle \text{vol}_M = \int_M (S_\alpha^* \Upsilon_\alpha \Phi_\alpha) \Psi_\alpha \text{vol}_M,$$

for every $\Phi_\alpha \in \text{Sol}(S_\alpha^\dagger)$ and $\Psi_\alpha \in \Gamma_c(E_\alpha)$. Therefore, Equation (3.6) rewrite as

$$S_1^* \Upsilon_1 R_\rho^\dagger = \Upsilon_1 S_1^\dagger R_\rho^\dagger = \Upsilon_1 \kappa_{1,0}^{\rho*} S_0^\dagger = \Upsilon_1 \kappa_{1,0}^{\rho*} \Upsilon_0^{-1} \Upsilon_0 S_0^\dagger = \kappa_{1,0}^{\rho*} S_0^* \Upsilon_0.$$

Since for any $\Psi_0 \in \text{Sol}(S_0^\dagger)$, it holds $\Upsilon_0 \Psi_0 \in \text{Sol}(S_0^*)$, we can conclude. \square

3.1 Conservation of Hermitian structures

In this section, we are going to show that the intertwining operator R_ϱ preserves Hermitian structures on the spaces of *homogeneous solutions with spacially compact support*, once that $\varrho \in C^\infty(M)$ is chosen suitably. Despite our result can be formulated abstractly for a generic symmetric hyperbolic system, we believe that analyzing the conservation of symplectic forms (for waves-like fields) and Hermitian scalar products (for Dirac fields) separately is more preparatory for Section 4.

3.1.1 Hermitian scalar products for Dirac fields

As already underlined in Remark 2.15, the space of spinors depends on the metric of the underlying manifold M_α . Therefore, an identification between spaces of sections of spinor bundles for different metrics is needed to construct an intertwining operator. This can be achieved by following [3, Section 5].

Let be $\lambda \in \mathbb{R}$, $g_\lambda \in \mathcal{GH}_M$ and consider a family of globally hyperbolic manifolds with the same Cauchy temporal function $M_\lambda := (M, g_\lambda)$. Let Z be the Lorentzian manifold

$$Z = \mathbb{I} \times M \quad g_Z = d\lambda^2 + g_\lambda.$$

On Z there exists a globally defined vector field which we denote as $e_\lambda := \frac{\partial}{\partial \lambda}$. For any λ , the spin structures on Z and M_λ are in one-to-one correspondence: Any spin structure on Z can be restricted to a spin structure on M_λ and a spin structure on M_λ it can be pulled back on Z – see [3, Section 3 and 5]. Actually, the spinor bundle SM_λ on each globally hyperbolic spin manifold M_λ can be identified with the restriction of the spinor bundle SZ on M_λ , in particular $SM_\lambda \simeq SZ|_{M_\lambda}$ if n is even, while $SM_\lambda \simeq S^+Z|_{M_\lambda} \simeq S^-Z|_{M_\lambda}$ if n is odd. Equivalently we may identify

$$SZ|_{\Sigma_t} = \begin{cases} SM_\lambda & \text{if } n \text{ is even,} \\ SZ|_{M_\lambda} \oplus SZ|_{M_\lambda} & \text{if } n \text{ is odd.} \end{cases} \quad (3.7)$$

By denoting with γ_Z the Clifford multiplication on SZ , the family of Clifford multiplications γ_λ satisfies satisfy

$$\gamma_\lambda(v)\psi = \gamma_Z(e_\lambda)\gamma_Z(v)\psi \quad \text{if } n \text{ is even,} \quad (3.8)$$

$$\gamma_\lambda(v)(\psi_+ + \psi_-) = \gamma_Z(e_\lambda)\gamma_Z(v)(\psi_+ - \psi_-) \quad \text{if } n \text{ is odd,} \quad (3.9)$$

where in the second case $\psi = \psi_+ + \psi_- \in SZ|_{M_\lambda} \oplus SZ|_{M_\lambda}$ and each component ψ_\pm is identified with an element in $S^\pm Z|_{M_\lambda}$.

Lemma 3.7. *Let Z be the Lorentzian manifold given by*

$$Z = \mathbb{I} \times M \quad g_Z = d\lambda^2 + g_\lambda,$$

where $(M, g_\lambda) := M_\lambda$ is a family of globally hyperbolic manifolds with $g_\lambda \in \mathcal{GH}_M$ for any $\lambda \in \mathbb{R}$, and denote with SM_λ be the spinor bundle over M_λ . For any $p \in M_\lambda$, the map

$$\kappa_{\lambda_1, \lambda_0} : S_p M_{\lambda_0} \rightarrow S_p M_{\lambda_1}. \quad (3.10)$$

defined by the parallel translation on Z along the curve $\lambda \mapsto (\lambda, p)$ is a linear isometry and preserves the Clifford multiplication.

Proof. Let $\psi_\alpha \in \Gamma(SZ)$ for $\alpha = 1, 2$ be parallel transported along the curve $\lambda \mapsto (\lambda, p)$, i.e. $\nabla_{e_\lambda} \psi_\alpha = 0$. Since the spin connection preserves the spin product, it follows that

$$\frac{d}{d\lambda} \langle \psi_1(\lambda) | \psi_2(\lambda) \rangle = \langle \nabla_{e_\lambda} \psi_1(\lambda) | \psi_2(\lambda) \rangle + \langle \psi_1(\lambda) | \nabla_{e_\lambda} \psi_2(\lambda) \rangle = 0.$$

Therefore the spin product on SZ is constant along the curve $\lambda \mapsto (\lambda, p)$ which implies that κ is an isometry. We conclude by showing that the Clifford multiplication is preserved. But this follows from the fact that e_λ and the Clifford multiplication γ_Z are parallel along $\lambda \mapsto (\lambda, p)$. Indeed, on account of the relations (3.8)-(3.9), we get

$$\begin{aligned} \kappa_{\lambda_1, \lambda_0} \left(\gamma_{\lambda_0}(u_{\lambda_0}) \psi_{\lambda_0} \right) &= \kappa_{\lambda_1, \lambda_0} \left(\gamma_Z(e_{\lambda_0}) \gamma_Z(u_{\lambda_0}) \psi_{\lambda_0} \right) = \\ &= \left(\gamma_Z(\kappa_{\lambda_1, \lambda_0} e_{\lambda_0}) \gamma_Z(\kappa_{\lambda_1, \lambda_0} u_{\lambda_0}) \kappa_{\lambda_1, \lambda_0} \psi_{\lambda_0} \right) = \gamma_{\lambda_1}(\kappa_{\lambda_1, \lambda_0} u_{\lambda_0}) (\kappa_{\lambda_1, \lambda_0} \psi_{\lambda_0}), \end{aligned}$$

so we can conclude our proof. \square

Let now D_α be the classical Dirac operator on a global hyperbolic manifold M_α . With the bundle isomorphism $\kappa_{\lambda_1, \lambda_0}$ we define the intertwining operator

$$R_\varrho : \text{Sol}_{sc}(D_0) \rightarrow \text{Sol}_{sc}(D_1)$$

where $\text{Sol}_{sc}(D_\alpha)$ is the space of homogeneous solutions with spatially compact support, *i.e.*

$$\text{Sol}_{sc}(D_\alpha) := \{ \psi_\alpha \in \Gamma_{sc}(SM_\alpha) \mid \psi_\alpha \in \ker D_\alpha \}.$$

Our next task is to show that, assigned M_0 and M_1 there exists a choice of ϱ such that R_ϱ preserves the positive definite Hermitian scalar product (3.11) naturally defined on $\text{Sol}_{sc}(D_\alpha)$. We begin by recasting the definition and an important property of the Hermitian scalar product.

Lemma 3.8 ([4], Lemma 3.17). *Let $\Sigma_\alpha \subset M_\alpha$ be a smooth spacelike Cauchy hypersurface with its future-oriented unit normal vector field \mathbf{n} and its induced volume element $\text{vol}_{\Sigma_\alpha}$. Then*

$$(\cdot | \cdot)_\alpha \text{Sol}_{sc}(D_\alpha) \times \text{Sol}_{sc}(D_\alpha) \rightarrow \mathbb{C} \quad (\psi_\alpha | \phi_\alpha)_\alpha = \int_\Sigma \langle \psi_\alpha | \gamma_\alpha(\mathbf{n}) \phi_\alpha \rangle \text{vol}_{\Sigma_\alpha}, \quad (3.11)$$

yields a positive definite Hermitian scalar product which does not depend on the choice of Σ_α .

Proposition 3.9. *Assume the setup of Theorem 3.4 and let $\varrho \in C^\infty(M)$ be such that $\text{vol}_{\Sigma_0} = \varrho^2 \text{vol}_{\Sigma_1}$, where $\text{vol}_{\Sigma_\alpha}$ is the volume induced by the metric g_α on a Cauchy hypersurface $\Sigma \subset J^-(\Sigma_-)$. Then the intertwining operator $R_\varrho : \text{Sol}_{sc}(D_0) \rightarrow \text{Sol}_{sc}(D_1)$ preserves the scalar products (3.11).*

Proof. On account of Proposition 2.16 and Theorem 3.4, for any $\tilde{\psi} := R^+ \kappa_{1,0}^\varrho \psi_0$ and $\tilde{\phi} := R^+ \kappa_{1,0}^\varrho \phi_0$, by Equation (3.4), it holds $R_- \tilde{\psi}, R_- \tilde{\phi} \in \text{Sol}_{sc}(D_1)$. Moreover, by Lemma 3.8, the scalar product does not depend on the choice of Σ , therefore

$$(R_- \tilde{\psi} | R_- \tilde{\phi})_1 = \int_\Sigma \langle R_- \tilde{\psi} | \gamma_1(\mathbf{n}) R_- \tilde{\psi} \rangle \text{vol}_\Sigma = \int_{\Sigma'} \langle R_- \tilde{\psi} | \gamma_1(\mathbf{n}) R_- \tilde{\psi} \rangle \text{vol}_{\Sigma'}.$$

By choosing $\Sigma' \subset J^+(\Sigma_+)$ we have $\chi = 1$, therefore the Hermitian scalar product reads as

$$(R_- \tilde{\psi} | R_- \tilde{\phi})_1 = \int_{\Sigma'} \langle \tilde{\psi} | \gamma_1(\mathbf{n}) \tilde{\psi} \rangle \text{vol}_{\Sigma'}.$$

On account of Equation (3.5), we get $\tilde{\psi}, \tilde{\phi} \in \text{Sol}_{sc}(D_{\chi,1}^\varrho)$. This implies that latter Hermitian form can be read as the scalar product $(\cdot | \cdot)_{\chi,1}$ on the space of solution $D_{\chi,1}^\varrho$, *i.e.*

$$(R_- \tilde{\psi} | R_- \tilde{\phi})_1 = \int_{\Sigma'} \langle \tilde{\psi} | \gamma_1(\mathbf{n}) \tilde{\psi} \rangle \text{vol}_{\Sigma'} = (\tilde{\psi} | \tilde{\phi})_{\chi,1}.$$

This because on Σ' , we have $\chi = 1$ which implies that $D_{\chi=1,1}^\varrho = D_1$ on small tubular neighborhood of Σ' contained in the future of Σ_+ . By Lemma 3.8, we thus obtain

$$(\tilde{\psi} | \tilde{\phi})_{\chi,1} = \int_{\Sigma'} \langle \tilde{\psi} | \gamma_{1,1}(\mathbf{n}) \tilde{\psi} \rangle \text{vol}_{\Sigma'} = \int_{\Sigma''} \langle \tilde{\psi} | \gamma_{\chi,1}(\mathbf{n}) \tilde{\psi} \rangle \text{vol}_{\Sigma''}.$$

By choosing $\Sigma'' = \Sigma$, which lies in the past of Σ_- , we obtain

$$(\tilde{\psi} | \tilde{\phi})_{\chi,1} = \int_{\Sigma} \langle \kappa_{1,0}^{\varrho} \psi_0 | \gamma_{0,1}(\mathbf{n}) \kappa_{1,0}^{\varrho} \psi_0 \rangle \text{vol}_{\Sigma} = (\kappa_{1,0}^{\varrho} \psi_0 | \kappa_{1,0}^{\varrho} \psi_0)_{0,1},$$

where the latter forms is defined on the solution space of the operator $D_{0,1}$. But, on account of Lemma 3.7, $\kappa_{1,0}$ is an isometry of spinor bundles which preserves the Clifford multiplication. Therefore it follows

$$(\kappa_{1,0}^{\varrho} \psi_0 | \kappa_{1,0}^{\varrho} \phi_0)_{0,1} = \int_{\Sigma} \langle \kappa_{1,0} \psi_0 | \gamma_{0,1}(\mathbf{n}) \kappa_{1,0} \phi_0 \rangle \varrho^2 \text{vol}_{\Sigma_1} = (\psi_0 | \phi_0)_0.$$

Plugging all together we can conclude. \square

3.1.2 Symplectic structures for geometric wave operator

Let M_{α} be globally hyperbolic manifolds with the same Cauchy temporal function and let P_{α} be a normally hyperbolic operator acting on section of a vector bundle V_{α} with finite rank and with a positive definite Hermitian form $\langle \cdot | \cdot \rangle_{\alpha}$. On account of Section 2.1.2, P_{α} can be reduced to a symmetric hyperbolic operator S_{α} acting on sections of the vector bundle $E_{\alpha} = V_{\alpha} \oplus (\mathbb{T}^* \Sigma \otimes V_{\alpha}) \oplus V_{\alpha}$, and the map $\iota : \text{Sol}(P_{\alpha}) \rightarrow \text{Sol}(S_{\alpha})$ is injective. We denote with $\text{Sol}_{P_{\alpha}}(S_{\alpha})$ the subspace of $\text{Sol}(S_{\alpha})$ where ι is bijective, *i.e.* $\text{Sol}_{P_{\alpha}}(S_{\alpha}) := \iota(\text{Sol}(P_{\alpha}))$.

Proposition 3.10. *Let $\alpha \in \{0, 1\}$ and consider P_{α} and S_{α} as above. If there exists a linear isometry $\bar{\kappa}_{1,0} : V_0 \rightarrow V_1$, then there exists an intertwining operator $\bar{R}_{\varrho} := \text{Sol}(P_0) \rightarrow \text{Sol}(P_1)$ which implements a non-canonical isomorphism.*

Proof. Since the tangent bundle does not depend on the underline metric, then we can define $\kappa_{1,0} : E_0 \rightarrow E_1$ as

$$\begin{pmatrix} f_0 \\ v \otimes f'_0 \\ f''_0 \end{pmatrix} \mapsto \begin{pmatrix} \bar{\kappa}_{1,0} f_0 \\ v \otimes \bar{\kappa}_{1,0} f'_0 \\ \bar{\kappa}_{1,0} f''_0 \end{pmatrix}.$$

Since $\bar{\kappa}_{1,0}$ is an isometry, then also $\kappa_{1,0}$ enjoys the same property. Hence the operator \bar{R}_{ϱ} which makes the following diagram commutative implements the desired non-canonical isomorphism

$$\begin{array}{ccc} \text{Sol}(P_0) & \xrightarrow{\bar{R}_{\varrho}} & \text{Sol}(P_1) \\ \downarrow \iota & & \downarrow \iota \\ \text{Sol}_{P_0}(S_0) & \xrightarrow{R_{\varrho}} & \text{Sol}_{P_1}(S_1) \end{array}$$

where R_{ϱ} is given as in Theorem 3.4. \square

Corollary 3.11. *Assume the setup of Theorem 3.4 and Proposition 3.10. The operator \bar{R}_{ϱ} reads as:*

$$\bar{R}_{\varrho} := \bar{R}_- \circ \bar{R}_+ \circ \bar{\kappa}_{1,0}^{\varrho}$$

where

$$\bar{R}_- := \text{Id} - G_1^-(1 - \chi)(P_1 - P_{0,1}^{\varrho}) \quad \text{and} \quad \bar{R}_+ := \text{Id} - G_{\chi}^+ \chi (P_1 - P_{0,1}^{\varrho})$$

where G_{χ}^+ is the advanced Green operator for $P_{\chi,1}^{\varrho}$ and G_1^- is the retarded Green operator for P_1 .

Proof. We begin by noticing that the principal symbols of $P_{\chi,1}^{\varrho} := (1 - \chi) P_{0,1}^{\varrho} + \chi P_1$ it satisfies

$$\begin{aligned} \sigma_{P_{\chi,1}^{\varrho}}(df)u_1 &= \frac{1}{2} P_{\chi,1}^{\varrho}(f^2)u_1 + \frac{1}{2} f^2 P_{\chi,1}^{\varrho}(u_1) = (1 - \chi) \sigma_{P_{0,1}^{\varrho}}(df)u_1 + \chi \sigma_{P_1}(df)u_1 \\ &= - \left((1 - \chi) g_0(df, df) + \chi g_1(df, df) \right) u_1, \end{aligned}$$

where we used that $\sigma_{P_{0,1}^\rho} = \bar{\kappa}_{1,0}^\rho \sigma_{P_0} \bar{\kappa}_{0,1}^\rho$, see e.g. Equation 3.2. By defining $h := (1 - \chi)g_0 + \chi g_1$, computations analogous to the one in Section 2.1.2, shows that $P_{\chi,1}^\rho$ can be reduced to a symmetric hyperbolic system. This is enough to guaranteed the existence of solutions and hence Green operators for $P_{\chi,1}^\rho$ as well. Using analogous computations to the ones performed in the proof of Theorem 3.4 we can already conclude. \square

As for Section 3.1.1, let $\text{Sol}_{sc}(P_\alpha)$ be the space of homogeneous solutions with spacially compact support, *i.e.*

$$\text{Sol}_{sc}(P_\alpha) := \{u_\alpha \in \Gamma_{sc}(V_\alpha) \mid u_\alpha \in \ker P_\alpha\}.$$

Our next task is to show that, assigned M_0 and M_1 there exists a choice of ρ such that \bar{R}_ρ preserves the symplectic form naturally defined on $\text{Sol}_{sc}(P_\alpha)$. As for the scalar product (3.11), also the symplectic form does not depends on the choice of the Cauchy hypersurface.

Lemma 3.12 ([4], Lemma 3.17). *Let $\Sigma_\alpha \subset M_\alpha$ be a smooth spacelike Cauchy hypersurface with its future-oriented unit normal vector field \mathbf{n} and its induced volume element $\text{vol}_{\Sigma_\alpha}$. Then*

$$\begin{aligned} \langle \cdot | \cdot \rangle_\alpha : \text{Sol}_{sc}(P_\alpha) \times \text{Sol}_{sc}(P_\alpha) &\rightarrow \mathbb{C} \\ \langle v_\alpha | v_\alpha \rangle_\alpha &= \int_\Sigma \langle v_\alpha | \nabla_{\mathbf{n}} u_\alpha \rangle - \langle u_\alpha | \nabla_{\mathbf{n}} v_\alpha \rangle \text{vol}_{\Sigma_\alpha}, \end{aligned} \quad (3.12)$$

yields a symplectic form which does not depend on the choice of Σ_α .

Using the same arguments as in the proof of Proposition 3.9 we can conclude an analog conservation of the symplectic form.

Proposition 3.13. *Assume the setup of Corollary 3.11 and let $\rho \in C^\infty(M)$ be such that $\text{vol}_{\Sigma_0} = \rho^2 \text{vol}_{\Sigma_1}$, where $\text{vol}_{\Sigma_\alpha}$ is the volume induced by the metric g_α on a Cauchy hypersurface $\Sigma \subset J^-(\Sigma_-)$. Then the intertwining operator \bar{R}_ρ preserves the symplectic form (3.12).*

4 Applications

We conclude this paper with an application inspired by [23, 32]. In *loc. cit.*, the quantization of a free field theory is interpreted as a two-step procedure:

1. The first consists of the assignment to a physical system of a $*$ -algebra of observables which encodes structural properties such as causality, dynamics and the canonical commutation/anti-commutation relations.
2. The second step calls for the identification of an algebraic state, which is a positive, linear and normalized functional on the algebra of observables.

Remark 4.1. This quantization scheme goes under the name of *algebraic quantum field theory* and it is especially well-suited for formulating quantum theories also on manifold – see e.g. [18, 36] for textbook, to [4, 5, 10, 35, 45] for recent reviews, [13–16] for homotopical approaches and [19–21, 24–29, 32–34] for some applications.

As explained in details by Benini, Dappiaggi and Schenkel in [12], the space of observables for the inhomogeneous solution coincides with the one of the homogeneous solution plus an extra observable assigned to the particular solution $G^+ \mathbf{f}$, where \mathbf{f} is the source term and G^+ is the advanced Green operator for the inhomogeneous Cauchy problem. Therefore, in what follows we shall only consider the quantization of classical fields which satisfies an homogeneous Cauchy problem.

4.1 Algebra of Dirac fields

As in Section 3.1.1, let $\text{Sol}_{sc}(\mathbb{D})$ be the space of *homogeneous* solutions with spatially compact support of the Dirac operator endowed with the positive definite Hermitian scalar product (3.11).

Definition 4.2. We call *algebra of Dirac fields* the unital, complex $*$ -algebra \mathfrak{A} freely generated by the abstract elements $\Xi(\psi)$, with $\psi \in \text{Sol}_{sc}(\mathbb{D})$, together with the following relations for all $\psi, \phi \in \text{Sol}_{sc}(\mathbb{D})$ and $\alpha, \beta \in \mathbb{C}$:

(i) Linearity: $\Xi(\alpha\psi + \beta\phi) = \alpha\Xi(\psi) + \beta\Xi(\phi)$

(ii) Hermiticity: $\Xi(\psi)^* = \Xi(\Upsilon\psi)$

(iii) Canonical anti-commutation relations (CARs):

$$\Xi(\psi) \cdot \Xi(\phi) + \Xi(\phi) \cdot \Xi(\psi) = 0 \quad \text{and} \quad \Xi(\psi) \cdot \Xi(\phi)^* + \Xi(\phi)^* \cdot \Xi(\psi) = (\psi | \phi) \text{Id}_{\mathfrak{A}}$$

where Υ is the adjunction map 2.6 and $(\cdot | \cdot)$ is the Hermitian scalar product (3.11).

A more concrete construction can be obtained as follow. Denote with

$$\text{Sol}_{\oplus} := \text{Sol}_{sc}(\mathbb{D}) \oplus \text{Sol}_{sc}(\mathbb{D}^*)$$

and consider the tensor \mathbb{C} -algebra $\mathfrak{T} := \left(\bigoplus_{n \in \mathbb{N}} (\text{Sol}_{\oplus})^{\otimes n}, \bullet \right)$. Notice that, on account of Remark 2.18, $\text{Sol}_{sc}(\mathbb{D}^*) = \Upsilon \text{Sol}_{sc}(\mathbb{D})$. The generators of \mathfrak{T} are given by

$$\text{Id}_{\mathfrak{T}} = \{1, 0 \dots\} \quad \Xi(\psi) := \left\{ 0, \begin{pmatrix} \psi \\ 0 \end{pmatrix}, 0 \dots \right\} \quad \Xi(\phi)^* := \left\{ 0, \begin{pmatrix} 0 \\ \Upsilon\phi \end{pmatrix}, 0 \dots \right\}$$

for any $\psi, \phi \in \text{Sol}_{sc}(\mathbb{D})$ and the involution $*$: $\mathfrak{T} \rightarrow \mathfrak{T}$ is implemented by means of the antilinear isomorphism Υ

$$\left\{ 0, \dots, \begin{pmatrix} \psi_1 \\ \Upsilon\phi_1 \end{pmatrix} \otimes \dots \otimes \begin{pmatrix} \psi_k \\ \Upsilon\phi_k \end{pmatrix}, 0 \dots \right\}^* = \left\{ 0, \dots, \begin{pmatrix} \phi_k \\ \Upsilon\psi_k \end{pmatrix} \otimes \dots \otimes \begin{pmatrix} \phi_1 \\ \Upsilon\psi_1 \end{pmatrix}, 0 \dots \right\}$$

for every $\psi_1, \phi_1, \dots, \psi_k, \phi_k \in \text{Sol}_{sc}(\mathbb{D})$. As always, $*$ is extended to all elements of \mathfrak{T} by antilinearity, thus turning \mathfrak{T} into a unital complex $*$ -algebra. The canonical relations are implemented taking the quotient of \mathfrak{T} by the $*$ -ideal \mathfrak{I} generated by

$$\Xi(\psi) \bullet \Xi(\phi) + \Xi(\phi) \bullet \Xi(\psi) \quad \text{and} \quad \Xi(\psi) \bullet \Xi(\phi)^* + \Xi(\phi)^* \bullet \Xi(\psi) - (\psi | \phi) \text{Id}_{\mathfrak{A}} \quad (4.1)$$

for every $\psi, \phi \in \text{Sol}_{sc}(\mathbb{D})$. It follows that $\mathfrak{A} = \mathfrak{T}/\mathfrak{I}$ is a realization of the algebra of Dirac fields.

Keeping in mind this concrete realization, we can prove the following isomorphism.

Theorem 4.3. *Assume the setup Proposition 3.9 and let \mathfrak{A}_{α} be the algebra of Dirac fields on M_{α} . Then there exists a $*$ -isomorphism $\mathfrak{A}_{1,0} : \mathfrak{A}_0 \rightarrow \mathfrak{A}_1$.*

Proof. Theorem 3.4 and Proposition 3.6 establish via R_{ρ} and R_{ρ}^* an isomorphism between $\text{Sol}(D_{\alpha})$ and $\text{Sol}(D_{\alpha}^*)$ respectively. As a by-product, $R_{\rho} \oplus R_{\rho}^*$ extends first of all to an isomorphism between the tensor algebras $\mathfrak{T}(\text{Sol}_{\alpha}^{\oplus})$ by linearity. Finally, on account of Proposition 3.9 $R_{\rho}^* = \Upsilon_1 R_{\rho} \Upsilon_0$ and R_{ρ} preserve the Hermitian scalar product, which implies that the ideals \mathfrak{I}_{α} are $*$ -isomorphic. \square

Before concluding this subsection, we want to make the following remark:

Remark 4.4. The algebra of Dirac fields cannot be considered an algebra of observables, since observables are required to commute at spacelike separations and \mathfrak{A} does not fulfil such requirement. However, the subalgebra $\mathfrak{A}_{\text{obs}} \subset \mathfrak{A}$ composed by even elements, *i.e.* $\Xi(\psi) = -\Xi(\psi)$, which are invariant under the action of $\text{Spin}_0(1, n)$ (extended to \mathfrak{A}) is a good candidate as algebra of observables. For further details we refer to [25, 30].

4.2 Algebras of real scalar fields

Consider a normally hyperbolic operator P acting on a real line bundle $V := \mathbb{R} \times M$. As in Section 3.1.2, let $\text{Sol}_{sc}(P)$ be the space of *homogeneous* solutions with spatially compact support endowed with the symplectic form (3.11).

Definition 4.5. We call *algebra of real scalar fields* the unital, complex $*$ -algebra \mathfrak{A} freely generated by the abstract elements $\Phi(u)$, with $u \in \text{Sol}_{sc}(P)$, together with the following relations for all $u, v \in \text{Sol}_{sc}(P)$ and $\alpha, \beta \in \mathbb{C}$:

(i) Linearity: $\Phi(\alpha u + \beta v) = \alpha \Phi(u) + \beta \Phi(v)$

(ii) Hermiticity: $\Phi(u)^* = \Phi(u)$

(iii) Canonical commutation relations (CCRs):

$$\Phi(v) \cdot \Phi(u) - \Phi(u) \cdot \Phi(v) = \langle v | u \rangle \text{Id}_{\mathfrak{A}}$$

where $\langle \cdot | \cdot \rangle$ is the symplectic form (3.11).

A more concrete construction can be obtained by mimicking the one for the Dirac fields. First, consider the tensor \mathbb{C} -algebra $\mathfrak{T} := \left(\bigoplus_{n \in \mathbb{N}} \text{Sol}_{sc}(P)^{\otimes n}, \bullet \right)$, where the generators of \mathfrak{T} are given by

$$\text{Id}_{\mathfrak{T}} = \{1, 0 \dots\} \quad \Phi(u) := \{0, u, 0 \dots\}$$

for any $u \in \text{Sol}_{sc}(D)$ and the involution $*$: $\mathfrak{T} \rightarrow \mathfrak{T}$ is implemented by means of the antilinear isomorphism Υ

$$\{0, \dots, u_1 \otimes \dots \otimes u_k, 0 \dots\}^* = \{0, \dots, u_k \otimes \dots \otimes u_1, 0 \dots\}$$

for every $\psi_1, \phi_1, \dots, \psi_k, \phi_k \in \text{Sol}_{sc}(D)$. As always, $*$ is extended to all elements of \mathfrak{T} by antilinearity, thus turning \mathfrak{T} into a unital complex $*$ -algebra. The canonical commutation relations are implemented taking the quotient of \mathfrak{T} by the $*$ -ideal \mathfrak{I} generated by

$$\Phi(u) \bullet \Phi(v) - \Phi(v) \bullet \Phi(u) - \langle u | v \rangle \text{Id}_{\mathfrak{A}} \tag{4.2}$$

for every $u, v \in \text{Sol}_{sc}(P)$. It follows that $\mathfrak{A} = \mathfrak{T}/\mathfrak{I}$ is a realization of the algebra of real scalar fields.

Keeping in mind this concrete realization, we can prove the following isomorphism.

Theorem 4.6. *Assume the setup Proposition 3.13 and let \mathfrak{A}_α be the algebra of real scalar fields on M_α . Then there exists a $*$ -isomorphism $\mathfrak{R}_{1,0} : \mathfrak{A}_0 \rightarrow \mathfrak{A}_1$.*

Proof. Corollary 3.11 establish via \bar{R}_Q an isomorphism between $\text{Sol}(P_\alpha)$. As a by-product, such result extends first of all to an isomorphism between the tensor algebras $\mathfrak{T}(\text{Sol}_{sc}(P_\alpha))$ by linearity. Finally, by Proposition 3.13 \bar{R}_Q preserves the Hermitian scalar product, which implies that the ideals \mathfrak{I}_α are $*$ -isomorphic. \square

4.3 Hadamard states

We conclude this section by studying (algebraic) states and their interplay with the intertwining operator \mathfrak{R} .

Definition 4.7. Given a complex $*$ -algebra \mathfrak{A} we call (*algebraic*) *state* any linear functional from \mathfrak{A} into \mathbb{C} that is positive, *i.e.* $\omega(\mathfrak{a}^* \mathfrak{a}) \geq 0$ for any $\mathfrak{a} \in \mathfrak{A}$, and normalized, *i.e.* $\omega(\text{Id}_{\mathfrak{A}}) = 1$.

Let V be a vector bundle and consider a Green hyperbolic operator $Q : V \rightarrow V$. Let $\text{Sol}_{sc}(Q)$ be the space of homogeneous solutions with spatially compact support.

Definition 4.8. Following Section 4.1 and Section 4.2 we call *algebra of fields*

$$\mathfrak{A} = \frac{\bigoplus_{n \in \mathbb{N}} \text{Sol}_{sc}(\mathbf{Q})^{\otimes n}}{\mathfrak{I}}$$

where \mathfrak{I} is a suitable $*$ -ideal which encodes CCR- or CAR-relations.

Remark 4.9. Notice that if we set $\mathbf{Q} = \mathbf{P} : C^\infty(\mathbf{M}) \rightarrow C^\infty(\mathbf{M})$ and we consider the ideal generated by the CCR relations (4.2), we thus obtain the algebra of real scalar fields. While if we set $\mathbf{Q} = \mathbf{D} \oplus \mathbf{D}^* : \Gamma(\mathbf{S}\mathbf{M}) \oplus \Gamma(\mathbf{S}^*\mathbf{M}) \rightarrow \Gamma(\mathbf{S}\mathbf{M}) \oplus \Gamma(\mathbf{S}^*\mathbf{M})$ and we consider the ideal generated by the CAR relations (4.1), we thus obtain the algebra of Dirac fields.

Due to the natural grading on the algebra of fields \mathfrak{A} , it suffices to ω on the monomials. This gives rise to the n -points distributions $\omega^{(n)} \in (\Gamma_c(\mathbf{V}))'$ by means of the relations

$$\omega^{(n)}(f_1, \dots, f_n) = \omega(f_1 \otimes \dots \otimes f_n) : \frac{(\text{Sol}_{sc}(\mathbf{Q}))^{\otimes n}}{\mathfrak{I}} \rightarrow \mathbb{C}. \quad (4.3)$$

where $u_j = \mathbf{G}f_j$, with $j = 1, \dots, n$, and \mathbf{G} is the causal Green propagators for \mathbf{Q} . This leads us to the following definition.

Definition 4.10. A state ω on the algebra of fields is *quasifree* if its n -point functions $\omega^{(n)}$ vanish for odd n , while for even n , they are defined as

$$\omega^{(n)}(f_1, \dots, f_n) = \sum_{\sigma \in S'_n} (-1)^{\text{sign}(\sigma)} \prod_{i=1}^{n/2} \omega^{(2)}(f_{\sigma(2i-1)}, f_{\sigma(2i)}),$$

where S'_n denotes the set of ordered permutations of n elements.

Remark 4.11. It is widely accepted that, among all possible (quasifree) states, the physical ones are required to satisfy the so-called *the Hadamard condition*. The reasons for this choice are manifold: For example, it implies the finiteness of the quantum fluctuations of the expectation value of every observable and it allows us to construct Wick polynomials following a covariant scheme, see [46] or [48] for recent reviews. This requirement is conveniently translated in the language of microlocal analysis, in particular into a microlocal characterization of the two-points distribution of the state. Since a full characterization is out of the scope of the paper, for further details we refer to [21, 37–39] for scalar fields and to [25, 33] for Dirac fields – see also [9, 11, 40, 51] for gauge theory.

With the next theorem, we show that the pull-back of a quasifree state along the isomorphism $\mathfrak{R}_{1,0} : \mathfrak{A}_0 \rightarrow \mathfrak{A}_1$ induced by the intertwining operator \mathbf{R}_ϱ for \mathbf{Q} (see e.g. Theorem 4.3 or Theorem 4.6) preserves singularity structure of the two-point distribution $\omega^{(2)}$, *i.e.* it preserves the wavefront set.

Theorem 4.12. *Assume the Setup 3.2 and denote with \mathfrak{A}_α the algebra of fields on \mathbf{M}_α . Finally let $\omega_\alpha : \mathfrak{A}_\alpha \rightarrow \mathbb{C}$ be quasifree states satisfying*

$$\omega_0 = \omega_1 \circ \mathfrak{R}_{1,0} : \mathfrak{A}_0 \rightarrow \mathbb{C}$$

with $\mathfrak{R}_{1,0}$ is the isomorphism induced by \mathbf{R}_ϱ . Then the bi-distributions of the associated two-point distributions $\omega_\alpha^{(2)}$ have the same singularity structure.

Proof. Since $\mathfrak{R}_{1,0}$ is $*$ -isomorphism, ω_0 inherits the property of being a quasifree state from ω_1 . In particular two-point function $\omega_0^{(2)}$ satisfies

$$\omega_0^{(2)}(f_0, g_0) := \omega_0(u_0 \otimes v_0) = \omega_1^{(2)}(\mathbf{R}_\varrho u_0 \otimes \mathbf{R}_\varrho v_0).$$

Consider now the restriction of $\omega_1^{(2)}$ to a neighborhood \mathbf{N}_1 of a Cauchy hypersurface $\Sigma_1 \subset J^+(\Sigma_+)$. On account of Theorem 4.12, $\chi = 1$ and we have $\mathbf{R}_\ell|_{\mathbf{N}_1} = \mathbf{R}_-\kappa_{1,0}^\ell$. In particular, ω_0 reads as

$$\omega_0^{(2)}(f_0, g_0)|_{\mathbf{N}_1} = \omega_1 \left(\mathbf{R}_-\kappa_{1,0}^\ell u_0 \otimes \mathbf{R}_-\kappa_{1,0}^\ell v_0 \right) \Big|_{\mathbf{N}_1}.$$

The latter two-points distribution can also be read as the restriction to \mathbf{N} of the two-point distribution ω_χ of a state on the algebra of \mathbf{Q}_χ -fields (*cf.* Proof of Theorem 3.4), namely

$$\omega_\chi : \mathfrak{A}_\chi \rightarrow \mathbb{C} \quad \omega_\chi \Big|_{\mathbf{N}_1} = \left(\omega_1 \circ \mathfrak{R}_{1,0} \right) \Big|_{\mathbf{N}_1}.$$

On account of the Theorem on the propagation of singularities, see e.g. [50], $\omega_\chi^{(2)}$ has the same singularities in the whole manifold. By restricting ω_χ to a neighborhood \mathbf{N}_2 of a Cauchy hypersurface $\Sigma_2 \subset J^-(\Sigma_-)$ we thus obtain

$$\omega_\chi^{(2)} \Big|_{\mathbf{N}_2} = \omega_1^{(2)} \left(\kappa_{1,0}^\ell f_0, \kappa_{1,0}^\ell g_0 \right) \Big|_{\mathbf{N}_2}.$$

Since the composition with a smooth map does not change the wavefront set, $\omega_0^{(2)}$ has the same singularity structure of $\omega_1^{(2)}$. \square

Remark 4.13. The main drawback of the definition of the intertwining operator \mathfrak{R} , used in Theorem 4.12, is the lack of any control on the action of the group of *-automorphism induced by the isometry group of \mathbf{M} on ω_2 . Let us remark, that the study of invariant states is a well-established research topic (*cf.* [7, 8]). Indeed, the type of factor can be inferred by analyzing which and how many states are invariant. From a more physical perspective instead, invariant states can represent equilibrium states in statistical mechanics e.g. KMS-states or ground states.

The previous remark leads us to the following open question: Under which conditions it is possible to perform an adiabatic limit, namely when $\lim_{\chi \rightarrow 1} \omega_1$ is well-defined?

A priori we expect that there is no positive answer in all possible scenarios, since it is known that certain free-field theories, e.g., the massless and minimally coupled (scalar or Dirac) field on four-dimensional de Sitter spacetime, do not possess a ground state, even though their massive counterpart does. A partial answer is given in [23, 31] for the case of scalar field theory on globally hyperbolic manifolds. In those papers, it is investigated how to relate normally hyperbolic operators which differ from a smooth potential, e.g. massive and massless wave operators.

We conclude this paper with the following corollary, which is a new proof of the existence of Hadamard states on every globally hyperbolic manifold.

Corollary 4.14. *Let (\mathbf{M}, g) be a globally hyperbolic manifold and denote with \mathfrak{A} the algebra of fields. Then there exists a state $\omega : \mathfrak{A} \rightarrow \mathbb{C}$ which satisfies the Hadamard condition.*

Proof. Let Σ be a Cauchy hypersurface for \mathbf{M} and denote with \mathcal{O} a globally hyperbolic open neighborhood of Σ , namely \mathcal{O} is an open neighborhood of Σ in \mathbf{M} containing all causal curves for \mathbf{M} whose endpoints lie in \mathcal{O} . Notice that \mathcal{O} is a globally hyperbolic submanifold of \mathbf{M} and it posses an algebra of fields which we denote by $\mathfrak{A}_\mathcal{O}$. Consider now $g|_\mathcal{O}$, the restriction of g to \mathcal{O} and denote with g_u a static metric such that $g_u \leq g|_\mathcal{O}$ and $g_u, g \in \mathcal{GH}_\mathcal{O}$. On a static globally hyperbolic manifold (\mathcal{O}, g_u) one may construct a ground state ω_H for the algebra of fields $\mathfrak{A}_\mathcal{O}^u$ which can be shown to be Hadamard (*cf.* Remark 4.11). By denoting with $\mathfrak{R} : \mathfrak{A}_\mathcal{O} \rightarrow \mathfrak{A}_\mathcal{O}^u$ the intertwining operator between the algebra of fields on $(\mathcal{O}, g|_\mathcal{O})$ and (\mathcal{O}, g_u) respectively (*cf.* Theorem 4.3 and Theorem 4.6), the state defined by

$$\omega = \omega_H \circ \mathfrak{R} : \mathfrak{A}_\mathcal{O} \rightarrow \mathbb{C}$$

satisfies the Hadamard condition on account of Theorem 4.12. On account of the time-slice axiom (see e.g. [10, Section 3]), $\mathfrak{A}_\mathcal{O}$ and \mathfrak{A} are *-isomorphic, so we can pull-back ω along the *-isomorphism and therefore we can conclude. \square

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