Spacelike deformations: Higher-spin fields from scalar fields

Vincenzo Morinelli *
Dipartimento di Matematica, Università di Roma Tor Vergata,
Via della Ricerca Scientifica, 1, I-00133 Roma, Italy
E-mail: morinell@mat.uniroma2.it

Karl-Henning Rehren
Institut für Theoretische Physik, Universität Göttingen,
37077 Göttingen, Germany
E-mail: rehren@theorie.physik.uni-goettingen.de

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Abstract

In contrast to Hamiltonian perturbation theory that changes the time evolution, "spacelike deformations" proceed by changing the translations (momentum operators). The free Maxwell theory is the first member of an infinite family of spacelike deformations of the complex massless Klein-Gordon quantum field into fields of higher spin.

1 Introduction

The basic idea of Hamiltonian perturbation theory is to start from a time zero algebra ("canonical commutation relations") equipped with a free time evolution, and perturb the free Hamiltonian such that the observables at later time $\Phi(t) := e^{iHt}\Phi_0 e^{-iHt}$ (where H is the perturbed Hamiltonian) deviate from the free ones. We present here a "complementary" deformation scheme for free quantum field theories: fixing the algebra along the time axis, we deform the space translations, so as to obtain a different local quantum field theory in Minkowski space. Similar ideas were previously pursued in two spacetime dimensional models [6].

Despite the apparent similarity, there are many differences, though. Hamiltonian Perturbation Theory (PT) is well-known to be obstructed by Haag's theorem, which implies that the perturbation is possible on the same Hilbert space only locally. Globally, the perturbed vacuum state is not a state in the "free Hilbert space", so that one is forced to change the representation of the time zero algebra. The need of renormalization of the mass also shows that one is even forced to change the time zero algebra itself. More precisely, interacting quantum fields in general do not even exist as distributions at a fixed time (see, e.g., [12, 13]).

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A recent approach [2], designed to avoid these obstructions, uses instead of a CCR time-zero algebra, an abstract "off-shell" C*-algebra of kinematical fields on spacetime that supports a large class of dynamics (one-parameter groups of time-evolution automorphisms). The invariant states under each dynamics, however, annihilate different ideals of the algebra ("field equations"), such that the corresponding GNS Hilbert spaces cannot be identified for any time-zero subalgebra.

In contrast, Wightman quantum fields can always be restricted to the time axis [1]. Our spacelike deformations are globally well-defined on the original Hilbert space. They consist in a redefinition of the generators of the spacelike translations (momentum operators). The perturbed fields away from the time axis are then defined as $\Phi(t,\vec{x}) = e^{ix^k \widetilde{P}_k}\Phi_0(t,0)e^{-ix^k \widetilde{P}_k}$, where $\widetilde{P}_k$ are the deformed generators.

In Hamiltonian PT, the "field content" is fixed by the choice of the free theory. The "particle content" is determined by the spectrum of the (renormalized) perturbed Hamiltonian, and may well change, e.g., when the interacting theory has bound states, or confinement occurs. Yet, the relation to the free particle content is usually not entirely lost.

In contrast, our spacelike deformations are (non-continuous) algebraic deformations that drastically change the field content without changing the Hamiltonian: e.g., one obtains the free Maxwell field by a deformation of a massless free scalar field.

In fact, we know spacelike deformations only for massless free fields, producing other massless free fields of any higher helicity (Sect. 2.3), or massive scalar fields (Sect. 3). The reason is that (a) we work on the one-particle space, from which the deformation passes to the Fock space by "second quantization"; and (b) we exploit conformal invariance in an essential way. The helicity deformations are in fact nonlinear deformations of the representation of the conformal Lie algebra, while the dilation operator enters the mass deformations.

We are therefore far from "interactions via deformation"; but our models illustrate the potential of a new approach, and more sophisticated new ideas may emerge from the present simple prototypes.

2 Helicity deformations

2.1 Background

The examples to be demonstrated rely on a recent observation in [7]: For the massless free fields of any integer helicity $h > 0$, the one-particle spaces $\mathcal{H}^{(h)}$ are strict subspaces of the one-particle space $\mathcal{H} = \mathcal{H}^{(0)}$ of the complex massless free scalar field. More precisely, $\mathcal{H}^{(h)}$ as representations $\widetilde{U}_h \oplus \widetilde{U}_{-h}$ of the Poincaré group extend to representations of the conformal group, whose restriction to the subgroup $\mathsf{M\ddot{o}b} \times \mathrm{SO}(3)$ is given by

$$\mathcal{H}^{(h)}|_{\mathsf{M\ddot{o}b}\times\mathsf{SO(3)}} = \bigoplus_{\ell=h}^{\infty} (\widetilde{U}^{(\ell+1)} \otimes \mathcal{D}^{(\ell)}) \oplus (\widetilde{U}^{(\ell+1)} \otimes \mathcal{D}^{(\ell)}),$$

where $\mathcal{D}^{(\ell)}$ are the spin-ℓ representations of SO(3), and $\widetilde{U}^{(d)}$ are the irreducible positive-energy representations of $\mathsf{M\ddot{o}b}$ with lowest eigenvalue d of L_0. The doubling is due to the "electric" and "magnetic" degrees of freedom. The same decomposition with $h=0$ holds for the complex scalar field, where the doubling corresponds to the subspaces of charge ± 1.

$\mathsf{M\ddot{o}b}\times \mathrm{SO}(3)$ is the subgroup of the conformal group that fixes the time axis $\vec{x}=0$. The vectors transforming in the displayed subrepresentations are spacelike derivatives of fields on the time axis, that transform like quasiprimary fields under $\mathsf{M\ddot{o}b}$, applied to the vacuum vector Ω. For the scalar field, these fields are simply [3]

$$P_\ell(\vec{\nabla})\varphi^{(*)}(x)|_{x=t,0}$$

with harmonic polynomials P_ℓ of degree ℓ, transforming like spin-ℓ multiplets of quasiprimary fields of scaling dimension $d=\ell+1$. For $h>0$, when the electric and magnetic fields are combined into a complex field tensor $F_{j_1\dots j_h}$, the equations of motion impose linear relations among the fields $\nabla_{i_1}\dots\nabla_{i_r}F^{(*)}_{j_1\dots j_h}(x)|_{x=t,0}$. One finds [7] exactly one quasiprimary spin-ℓ multiplet (both for F and F^*) of scaling dimension $\ell+1$ for each $\ell\geq h$.

In this count, as h increases, the field content decreases. The lowest fields of the scalar theory are given by $\varphi^{(*)}(t,\vec{0})$ ($\ell=0$, $d=1$) and $\vec{\nabla}\varphi^{(*)}(t,\vec{0})$ ($\ell=1$, $d=2$), while, e.g., the Maxwell theory starts at $\ell=1$, $d=2$ with the vector field $\vec{F}^{(*)}=\vec{E}\pm i\vec{B}$. In this sense, contrary to intuition, the higher-helicity theories have *less* degrees of freedom than the lower-h theories.

In [7], these facts were exploited to estimate the trace of $e^{-\beta L_0}$, whose finiteness then implies the split property for all finite-helicity massles free quantum field theories. Here, we take them as the starting point of spacelike deformation, as already speculated in [7].

To illustrate the idea, consider the case $h=1$ (Maxwell). The Maxwell equations for $\vec{F}$ read

$$\vec{\nabla}\cdot\vec{F}=0,\quad \vec{\nabla}\times\vec{F}=i\partial_t\vec{F}.$$

The component fields $F_k(t,\vec{0})$ on the time axis transform in the same way under Möbius transformations of the time axis and rotations, like the fields $\nabla_k\varphi(x,\vec{0})$ of the complex massless Klein-Gordon theory. Similarly, the spin-2 fields $(\nabla_iF_j+\nabla_jF_i)(t,\vec{0})$ transform in the same way as the fields $(\nabla_i\nabla_j-\frac{1}{3}\delta_{ij}\Delta)\varphi(t,\vec{0})$.

Because the representations of the Möbius and rotation groups on the one-particle spaces are the same – except for the absence of the subrepresentation with $\ell=0$ in the Maxwell theory – we can algebraically identify these pairs of fields along the time axis. We get

$$F_i(t,\vec{0})\stackrel{!}{=}2\nabla_i\varphi(t,\vec{0}) \tag{2.1}$$

$$\nabla_iF_j(t,\vec{0})\stackrel{!}{=}\alpha\cdot(\nabla_i\nabla_j-\frac{1}{3}\delta_{ij}\Delta)\varphi(t,\vec{0})+i\varepsilon_{ijk}\partial_t\nabla_k\varphi(t,\vec{0}), \tag{2.2}$$

where the Maxwell equations dictate the anti-symmetric part in (2.2) as well as the absence of an $\ell=0$ contribution; the normalizations are fixed via the two-point functions, giving $|\alpha|^2=12$.

The problem is apparent: the left-hand side of (2.2) is the derivative of the left-hand side of (2.1), which is not true for the right-hand sides. The spatial derivatives being implemented by the momentum operators P_k, we conclude that while the Möbius and rotation generators of both theories (including the Hamiltonian P_0) can be identified, their spatial momentum operators must differ.

We are going to determine the momentum operators $\widetilde{P}_k$ of the Maxwell theory as polynomials of the conformal and charge generators of the Klein-Gordon theory (and along with them the boosts and the generators of spatial special conformal transformations). Then, starting from the identification (2.1) as a definition of the Maxwell field on the time axis, and acting with $\widetilde{U}(\vec{x}) = e^{ix^k \widetilde{P}_k}$ on $\varphi(t, \vec{0})$, one obtains the Maxwell field everywhere in Minkowski space. The same works for any helicity $h > 0$.

As a second instance, we present the spacelike deformation of the massless scalar field into the massive scalar field in Sect. 3.

The mere existence of such deformations should not be too surprising, given that "all Hilbert spaces are the same". The noticeable facts are that the deformations fix parts of the symmetry, and that they can be given on the remaining generators by explicit formulae.

2.2 Preliminaries about the conformal Lie algebra

We denote by P_μ, $M_{\mu\nu}$, D, K_μ the generators of translations, Lorentz transformations, dilations, and special conformal transformations in the conformal Lie algebra $\mathfrak{so}(2,4)$, respectively. Their commutators are explicitly

$$\begin{aligned} &i[P_\mu, P_\nu] = 0, \quad &&i[P_\mu, M_{\kappa\lambda}] = \eta_{\mu\lambda}P_\kappa - \eta_{\mu\kappa}P_\lambda, \quad &&i[M_{\kappa\lambda}, M_{\mu\nu}] = \eta_{\kappa\mu}M_{\lambda\nu} \pm \ldots; \\ &i[D, P_\mu] = P_\mu, \quad &&i[D, K_\mu] = -K_\mu, \quad &&i[D, M_{\kappa\lambda}] = 0; \\ &i[K_\mu, K_\nu] = 0, \quad &&i[M_{\kappa\lambda}, K_\mu] = \eta_{\kappa\mu}K_\lambda - \eta_{\lambda\mu}K_\kappa, \quad &&i[P_\mu, K_\nu] = -2\eta_{\mu\nu}D + 2M_{\mu\nu}. \end{aligned} \tag{2.3}$$

In particular, we have the Lie subalgebras $\mathfrak{m\ddot{o}b}$:

$$i[D, P_0] = P_0, \quad i[P_0, K_0] = -2D, \quad i[D, K_0] = -K_0,$$

and $\mathfrak{so}(3)$:

$$i[M_{ij}, M_{kl}] = \delta_{jk}M_{il} - \delta_{jl}M_{ik} - \delta_{ik}M_{jl} + \delta_{il}M_{jk}.$$

The conformal transformations of the massless Klein-Gordon field are given by the infinitesimal action of $\mathfrak{so}(2,4)$:

$$\begin{aligned} &i[P_\mu, \varphi(x)] = \partial_\mu\varphi(x), \quad &&i[M_{\mu\nu}, \varphi(x)] = (x_\mu\partial_\mu - x_\nu\partial_\mu)\varphi(x), \\ &i[D, \varphi(x)] = ((x\partial) + 1)\varphi(x), \quad &&i[K_\mu, \varphi(x)] = (2x_\mu(x\partial) - x^2\partial_\mu + 2x_\mu)\varphi(x). \end{aligned} \tag{2.4}$$

Lemma 2.1. *The parity reflection $(t, \vec{x}) \mapsto (t, -\vec{x})$ defines a symmetric space decomposition of the conformal Lie algebra*

$$\mathfrak{so}(2,4) = \mathfrak{h} \oplus \mathfrak{m}, \quad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m} \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h},$$

where $\mathfrak{h} = \mathfrak{m\ddot{o}b} \oplus \mathfrak{so}(3) = \mathrm{Span}(P_0, D, K_0, M_{kl})$ and $\mathfrak{m} = \mathrm{Span}(P_k, M_{0k}, K_k)$. The generators of $\mathfrak{m}$ transform like vectors under $\mathfrak{so}(3)$:

$$i[M_{kl}, X_i] = \delta_{li}X_k - \delta_{ki}X_l, \tag{2.5}$$

and $\mathfrak{m\ddot{o}b}$ acts on $\mathfrak{m}$ as a $\mathfrak{m\ddot{o}b}$-module like

$$\begin{aligned} &i[P_0, P_k] = 0, \quad &&i[P_0, M_{0k}] = -P_k, \quad &&i[P_0, K_k] = 2M_{0k}, \\ &i[D, P_k] = P_k, \quad &&i[D, M_{0k}] = 0, \quad &&i[D, K_k] = -K_k, \\ &i[K_0, P_k] = 2M_{0k}, \quad &&i[K_0, M_{0k}] = -K_k, \quad &&i[K_0, K_k] = 0. \end{aligned} \tag{2.6}$$

In particular, $(\mathrm{ad}_{P_0})^3 = (\mathrm{ad}_{K_0})^3$ *on* $\mathfrak{m}$.

Proof: Immediate from (2.3). $\square$

We finally list the commutation relations of the conformal generators and the charge operator with the anti-unitary PCT operator J:

$$JP_\mu = P_\mu J, \quad JM_{\mu\nu} = -M_{\mu\nu}J, \quad JD = -DJ, \quad JK_\mu = K_\mu J, \quad JQ = -QJ. \tag{2.7}$$

2.3 Main result

Let $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ the one-particle space of the complex massless Klein-Gordon field, where the superscript $\pm$ stands for the eigenvalue ± 1 of the charge operator Q. As representations of $\mathsf{M\ddot{o}b} \times \mathrm{SO}(3)$, both $\mathcal{H}^\pm$ decompose as

$$\mathcal{H}^\pm|_{\mathsf{M\ddot{o}b}\times\mathsf{SO(3)}} = \bigoplus_{\ell=0}^{\infty} \mathcal{H}^\pm_\ell, \qquad \mathcal{H}^+_\ell \cong \mathcal{H}^-_\ell \cong \widetilde{U}^{(\ell+1)} \otimes \mathcal{D}^{(\ell)}. \tag{2.8}$$

Let E_ℓ be the projections onto the subspaces $\mathcal{H}_\ell = \mathcal{H}^+_\ell \oplus \mathcal{H}^-_\ell$, $E^{(h)} = \sum_{\ell \geq h}$, and $\mathcal{H}^{(h)} = E^{(h)}\mathcal{H} = \bigoplus_{\ell\geq h} \mathcal{H}_\ell$. Let P_μ, $M_{\mu\nu}$, D, K_μ the generators of the conformal Lie algebra (2.3) represented on the one-particle space of the complex massless Klein-Gordon field.

The main result defines deformed generators $\widetilde{P}_k$ (the translations of the deformed QFT) in terms of the generators of the scalar QFT on the subspace $\mathcal{H}^{(h)}$ of the one-particle space of the scalar QFT.

Proposition 2.2. *Let h a non-negative integer. For $k = 1, 2, 3$, define self-adjoint deformed generators $\widetilde{P}_k$, $\widetilde{M}_{0k}$, $\widetilde{K}_k$ of $\mathfrak{m}$ on $\mathcal{H}^{(h)}$ by*

$$\widetilde{P}_k := \sum\nolimits_{\ell\geq h} a_\ell \cdot (E_{\ell+1}P_kE_\ell + E_\ell P_k E_{\ell+1}) + \sum\nolimits_{\ell \geq h} b_\ell \cdot QS_kE_\ell, \tag{2.9}$$

$$2\widetilde{M}_{0k} := i[K_0, \widetilde{P}_k], \qquad -\widetilde{K}_k := i[K_0, \widetilde{M}_{0k}], \tag{2.10}$$

where $S_k := \varepsilon_{kmn}(P_mM_{0n} + M_{0n}P_m)$ commute with E_ℓ, and the coefficients a_ℓ, b_ℓ are real.

(i) The deformed generators $\widetilde{P}_k$, $\widetilde{M}_{0k}$, $\widetilde{K}_k$ satisfy the correct commutation relations (2.7) with the PCT operator.
(ii) Together with the undeformed generators P_0, D, K_0 of $\mathfrak{m\ddot{o}b}$ and M_{kl} of $\mathfrak{so}(3)$, they satisfy the conformal Lie algebra (2.3) on $\mathcal{H}^{(h)}$ if and only if

$$a_\ell^2 = \frac{(\ell+1)^2 - h^2}{(\ell+1)^2}, \qquad b_\ell^2 = \frac{h^2}{4\ell^2(\ell+1)^2} \tag{2.11}$$

and all coefficients b_ℓ have the same sign.
(iii) The generators $\widetilde{P}_k$ as specified by (ii) satisfy the mass-shell condition on $\mathcal{H}^{(h)}$:

$$\sum\nolimits_k \widetilde{P}_k^2 = P_0^2. \tag{2.12}$$

Proof: (i) is immediate by (2.7). Because S_k transform as a vector under $\mathfrak{so}(3)$, they can change ℓ by at most one. Inspection of the operators $\widetilde{P}_k$ and $\widetilde{M}_{0k}$ shows that they both change ℓ by exactly one (see below). Hence S_k commute with E_ℓ. To prove (ii), we start with a Lemma.

Lemma 2.3. *(i) The deformed generators (2.9), (2.10) fulfill the correct* $[\mathfrak{h},\mathfrak{m}]$ *commutation relations (2.5) and (2.6) independent of the specification of the coefficients.*
(ii) The remaining $[\mathfrak{m},\mathfrak{m}]$ *commutation relations are also true on* $\mathcal{H}_\ell$ $(\ell \geq h)$*, if and only if*

$$i[\widetilde{P}_k,\widetilde{K}_l] = 2\delta_{kl}D + 2M_{kl}. \tag{2.13}$$

Proof of the Lemma: (i) is obvious because the projections E_ℓ and the charge operator Q commute with $\mathfrak{h} = \mathfrak{m\ddot{o}b} \oplus \mathfrak{so}(3)$. (ii) follows by repeated application of ad_{K_0} and ad_{P_0} to (2.13), using (2.6) and (2.5). $\square$

(Alternatively, assuming the correct Poincaré commutation relations $i[\widetilde{P}_k,\widetilde{P}_l] = 0$, $i[\widetilde{M}_{0k},\widetilde{P}_l] = \delta_{kl}P_0$, $i[\widetilde{M}_{0k},\widetilde{M}_{0l}] = M_{kl}$, would also do the job.)

The remaining proof of the proposition is just a cumbersome direct computation with the generators sandwiched between projections. Starting from (2.4), we determine the action on (improper) time-dependent vectors $|j_1\dots j_s\rangle_t := \nabla_{j_1}\dots\nabla_{j_s}\varphi(t,\vec{x})\Omega$ of spin $\ell \leq s$ in the subspace $\mathcal{H}^+$ of charge $+1$.

The generators of the scalar QFT act as follows on the one-particle space. In a schematic notation,

$$\begin{aligned}
iP_0|j_1\dots j_s\rangle_t &= \partial_t|j_1\dots j_s\rangle_t, && (2.14a)\\
iD|j_1\dots j_s\rangle_t &= (t\partial_t+s+1)|\dots\rangle_t, && (2.14b)\\
iK_0|j_1\dots j_s\rangle_t &= (t^2\partial_t+2(s+1)t)|\dots\rangle_t + \partial_t\sum\nolimits_{a\neq b}\delta_{j_aj_b}|\cdots\times\times\rangle_t, && (2.14c)\\
iM_{kl}|j_1\dots j_s\rangle_t &= \sum\nolimits_a\big(\delta_{lj_a}|k\cdots\times\rangle_t - \delta_{kj_a}|l\cdots\times\rangle_t\big), && (2.14d)\\
iP_k|j_1\dots j_s\rangle_t &= |k\dots\rangle_t, && (2.14e)\\
iM_{0k}|j_1\dots j_s\rangle_t &= t|k\dots\rangle_t + \partial_t\sum\nolimits_a\delta_{kj_a}|\cdots\times\rangle_t, && (2.14f)\\
iK_k|j_1\dots j_s\rangle_t &= -t^2|k\dots\rangle_t - 2(t\partial_t+s)\sum\nolimits_a\delta_{kj_a}|\cdots\times\rangle_t + \sum\nolimits_{a\neq b}\delta_{j_aj_b}|k\cdots\times\times\rangle_t, && (2.14g)
\end{aligned}$$

where "$\times$" indicates the deletion of an index j_a or j_b. In order to pass to the (improper) spin-ℓ vectors $|j_1\dots j_\ell\rangle_t^\ell$, one has to subtract contractions of derivatives, which due to the wave equation $\Delta\varphi = \partial_t^2\varphi$ are given by

$$|j_1\dots j_\ell\rangle_t^\ell = |j_1\dots j_\ell\rangle_t - \frac{1}{2(2\ell-1)}\partial_t^2\sum\nolimits_{a\neq b}\delta_{j_aj_b}|\cdots\times\times\rangle_t \pm \dots, \tag{2.15}$$

where "$\dots$" stands for higher contractions. Thus, if $X_k = P_k$, M_{0k}, K_k are written in the form

$$iX_k|j_1\dots j_s\rangle_t = A|k\dots\rangle_t + B_s\sum\nolimits_a\delta_{kj_a}|\cdots\times\rangle_t + C\sum\nolimits_{a\neq b}\delta_{j_aj_b}|k\cdots\times\times\rangle_t$$

with differential operators A, B_s, C w.r.t. the time t as in (2.14e–g), then

$$iX_k|j_1\dots j_\ell\rangle_t^\ell = A|k\dots\rangle_t^{\ell+1} + B_\ell^0\sum\nolimits_a\delta_{kj_a}|\cdots\times\rangle_t^{\ell-1} + C_\ell^0\sum\nolimits_{a\neq b}\delta_{j_aj_b}|k\cdots\times\times\rangle_t^{\ell-1},$$

where $B_\ell^0 = B_\ell + \frac{1}{2\ell+1}A\partial_t^2$, $C_\ell^0 = C - \frac{1}{2(2\ell-1)}\partial_t^2A + \frac{1}{2(2\ell+1)}A\partial_t^2$. The higher contractions in (2.15) do not contribute because the vector operators X_k can change ℓ by at most one. We refrain from displaying the explicit formulae. They are exactly the same for the oppositely charged vectors $|j_1\dots j_s\rangle_t^* := \nabla_{j_1}\dots\nabla_{j_s}\varphi^*(t,\vec{0})\Omega$ and $|j_1\dots j_\ell\rangle_t^{*\ell}$ in $\mathcal{H}^-$.

From these expressions, it can be seen that the deformed generators make only transitions to spin $\ell+1$ (the A-term) and to $\ell-1$ (the B^0 and C^0 terms). One may then evaluate the commutator (2.13) on arbitrary vectors $|j_1\dots j_\ell\rangle_t^\ell$ and equate the result with the desired right-hand side. This gives several conditions on the coefficients a_ℓ and b_ℓ in (2.9), that turn out to be equivalent to the system

$$\begin{aligned} a_\ell^2 + 4\ell^2\cdot b_\ell^2 &= 1, & (2.16a)\\ a_\ell^2 - a_{\ell-1}^2 &= 4(2\ell+1)\cdot b_\ell^2, & (2.16b)\\ \ell\cdot a_\ell b_\ell &= (\ell+2)\cdot a_\ell b_{\ell+1}. & (2.16c)\end{aligned}$$

((2.16a) ensures the correct symmetric term $2\delta_{kl}D$ in $iE_\ell[\widetilde{P}_k,\widetilde{K}_l]E_\ell$, and together with (2.16a), (2.16b) ensures the correct anti-symmetric term $2M_{kl}$. (2.16c) ensures the absence of a spin-changing contribution.)

Eliminating b_ℓ^2 from (2.16a) and (2.16b), one gets a simple recursion

$$(\ell+1)^2 a_\ell^2 - \ell^2 a_{\ell-1}^2 = (\ell+1)^2 - \ell^2,$$

hence $(\ell+1)^2(a_\ell^2-1) = const$. The initial condition $a_{h-1}=0$ gives $const = -h^2$, hence (2.11). (2.16c) shows that b_ℓ have constant sign.

This proves (ii). Evaluation of (2.12) on arbitrary vectors $|j_1\dots j_\ell\rangle_t^\ell$ with the given values (2.11), yields the desired result, proving (iii). $\square$

The signs of b_ℓ may be chosen positive without loss of generality, via the unitary charge conjugation. Also the coefficients a_ℓ may all be chosen positive via a unitary involution in the center of $\mathfrak{m\ddot{o}b}\oplus\mathfrak{so}(3)$.

2.4 Field algebras

The construction of a local QFT on the Fock space $\mathcal{F}^{(h)} = \Gamma(\mathcal{H}^{(h)})$ over $\mathcal{H}^{(h)}$ is routine.

The deformed generators generate a unitary true representation $\widetilde{U}$ of the conformal group on $\mathcal{H}^{(h)}$ and (by second quantization) on $\mathcal{F}^{(h)}$, because the spectrum of L_0 on the subspace is a subset of the spectrum on the original Hilbert space $\mathcal{F}^{(0)}$, and hence is still integer.

For open intervals $I\subset\mathbb{R}$ (the time axis), let O_I be the corresponding doublecone spanned by I, and define $\widetilde{A}(O_I) := A(O_I)|_{\mathcal{F}^{(h)}}$, where $A(O)$ are the local algebras of the scalar field theory. For arbitrary doublecones, choose an interval I and a conformal transformation g such that $gO_I = O$, and define

$$\widetilde{A}(O) := \widetilde{U}(g)\widetilde{A}(O_I)\widetilde{U}(g)^*.$$

The definition is unambiguous because if $g_1O_{I_1} = O = g_2O_{I_2}$, then $g_2^{-1}g_1(I_1) = I_2$, hence $g := g_2^{-1}g_1 \in \mathsf{M\ddot{o}b}\times\mathrm{SO}(3)$, hence $\widetilde{U}(g)A(I_1)\widetilde{U}(g)^* = A(I_2)$. Thus, the net

$$O \mapsto \widetilde{A}(O)$$

is conformally covariant.

Because for any pair of spacelike separated doublecones, there is a conformal transformation g mapping the doublecones into the forward and backward lightcones, respectively, locality follows from the Huygens property of the scalar field ($A(V_+)$ commutes with $A(V_-)$) by covariance (see, e.g., [8]).

2.5 Field equations

At the level of fields, we identify (with the appropriate normalization factor) $F^{(*)}_{j_1\cdots j_h}(t,\vec{0})$ with the derivative fields $p_{j_1\ldots j_h}(\vec{\nabla})\varphi^{(*)}(t,\vec{0})$ (harmonic polynomials of spin $\ell = h$) restricted to $\mathcal{F}^{(h)}$, and hence $F^{(*)}_{j_1\cdots j_h}(t,\vec{0})\Omega$ with the (improper) spin-h vectors $|j_1\ldots j_h\rangle^{(*)h}_t$ of $\mathcal{H}^\pm_h$. We define $F^{(*)}_{j_1\cdots j_h}(t,\vec{x})$ by the adjoint action of $e^{ix_k\widetilde{P}_k}$ such that $F^{(*)}_{j_1\cdots j_h}(t,\vec{x})\Omega$ are improper vectors in $\mathcal{H}^{(h)\pm}$. Then $F^{(*)}(x)$ smeared within a doublecone O are affiliated with the algebra $\widetilde{A}(O)$, hence they are local fields.

In order to make the identification of the deformed field with the free helicity-h field, we have to establish the equation of motion [7]

$$\nabla_k F_{kj_2\ldots j_h} = 0, \qquad \varepsilon_{kjm}\nabla_k F_{jj_2\ldots j_h} = i\partial_t F_{mj_2\ldots j_h}, \tag{2.17}$$

where $F = E+iB$, and $E_{j_1\cdots j_h}$ and $B_{j_1\cdots j_h}$ are symmetric traceless "electric" and "magnetic" tensors. On the time axis, we have by construction (with the appropriate normalization)

$$F_{j_1\ldots j_h}(t,\vec{0})\Omega = c\,|j_1\ldots j_h\rangle^h_t, \quad \text{hence} \quad \nabla_j F_{j_1\cdots j_h}(t,\vec{0})\Omega = c\cdot i\widetilde{P}_j|j_1\ldots j_h\rangle^h_t.$$

With Prop. 2.2, we compute

$$i\widetilde{P}_k|j_1\ldots j_h\rangle^h_t = a_h|kj_1\ldots j_h\rangle^{h+1}_t + 2b_h\cdot i\sum\nolimits_a \varepsilon_{kj_am}\partial_t|m\cdots\times\rangle^h_t. \tag{2.18}$$

This immediately implies

$$\widetilde{P}_k|kj_2\ldots j_h\rangle^h_t = 0, \qquad \varepsilon_{kjm}\widetilde{P}_k|jj_2\ldots j_h\rangle^h_t = 2(h+1)b_h\cdot\partial_t|mj_2\ldots j_h\rangle^h_t. \tag{2.19}$$

Because $2(h+1)b_h = 1$, the higher Maxwell equations hold on the time axis and on the vacuum vector:

$$\nabla_j F_{jj_2\ldots j_h}(t,\vec{0})\Omega = 0, \qquad \varepsilon_{ijk}\nabla_i F_{jj_2\ldots j_h}(t,\vec{0})\Omega = i\partial_t F_{kj_2\ldots j_h}(t,\vec{0})\Omega. \tag{2.20}$$

The complex conjugate higher Maxwell equations for $F^* = E - iB$ are guaranteed by the presence of the operator Q in (2.9), that switches the sign of i in the right-hand side of (2.18) for the vectors of charge -1.

At this point, it becomes apparent how the charge of the scalar field is re-interpreted as the sign of the helicity of the higher Maxwell field.

By applying the spacelike translations $\widetilde{U}(\vec{x})$ to (2.20), we conclude that the higher Maxwell equations on the vacuum vector hold everywhere in Minkowski space. Because $F^{(*)}$ are local fields on the time axis, by conformal covariance they are local on Minkowski spacetime. Then the Reeh-Schlieder theorem ensures that the higher Maxwell equations hold as operator equations.

3 Mass deformation

A second, and much simpler, instance of spacelike deformation is the construction of the massive Klein-Gordon field as a deformation of the spacelike translations and boosts of the massless Klein-Gordon field. It is "complementary" to the corresponding Hamiltonian deformation treated in [5].

In this instance, we can just write down the deformed generators. Because the deformations turn out to concern the differential operators w.r.t. t, it will be advantageous to pass to the Fourier transform on the time axis: $|\dots\rangle_\omega = \int dt\, e^{-i\omega t}|\dots\rangle_t$. Because the massive one-particle vectors have energy $\omega \geq m$, the spectral projection $E_m = \theta(P_0 - m)$ will play the role of the projection $E^{(h)}$ in Sect. 2.3.

The Lie algebra of the Poincaré group again has a symmetric space decomposition $\mathfrak{h}\oplus\mathfrak{m}$, where $\mathfrak{h} = \mathfrak{t}\oplus\mathfrak{so}(3)$ ($\mathfrak{t}$ = time translations), and $\mathfrak{m}$ is spanned by the spacelike momenta and the boost generators. By definition, the rotations M_{kl} and the Hamiltonian P_0 remain undeformed. Thus, the deformed boosts $\widetilde{M}_{0k}$ also determine the deformed momenta $\widetilde{P}_k = -i[P_0, \widetilde{M}_{0k}]$.

We return to the real scalar field, and denote by φ_0 and φ_m the massless and massive fields. Whenever P, P' are homogeneous polynomials of degree s, s', the scalar product in spherical coordinates is

$$(P(\vec{\nabla})\varphi_m(t,\vec{0})\Omega, P'(\vec{\nabla})\varphi_m(t',\vec{0})\Omega) = i^{s'-s}\int \frac{p^2\,dp}{2p^0}\cdot p^{s+s'}\int d\sigma(\vec{n})\overline{P(\vec{n})}P'(\vec{n})\cdot e^{-ip_0(t-t')}, \quad (3.1)$$

where $p^0 = \sqrt{p^2+m^2}$, $\vec{p} = p\vec{n}$, and $d\sigma$ is the invariant measure on the sphere. Passing to the integration variable $\omega = p_0$, one has

$$\frac{p^{2+s+s'}dp}{2p^0} = \frac{1}{2}(\omega^2 - m^2)^{(s+s'+1)/2}d\omega.$$

If we write $|j_1\dots j_s\rangle_t := \nabla_{j_1}\dots\nabla_{j_s}\varphi_0(t,\vec{0})\Omega$ as before, and ${}^m|j_1\dots j_s\rangle_t := \nabla_{j_1}\dots\nabla_{j_s}\varphi_m(t,\vec{0})\Omega$, this shows that

$$U : {}^m|j_1\dots j_s\rangle_\omega \mapsto \sigma(\omega)^{s+\frac{1}{2}}|j_1\dots j_s\rangle_\omega, \quad \text{where} \quad \sigma(\omega) = \Big(1 - \frac{m^2}{\omega^2}\Big)^{\frac{1}{2}}, \quad (3.2)$$

is a unitary identification $U : \mathcal{H}_m \to E_m\mathcal{H}_0$ of the massive one-particle space $\mathcal{H}_m$ with the subspace $E_m\mathcal{H}_0$ of the massless one-particle space $\mathcal{H}_0$. The deformed Poincaré generators on $E_m\mathcal{H}_0$ arise by the unitary conjugation Ad_U of the known action of the massive Poincaré generators on $\mathcal{H}_m$, i.e., the pull-back under the identification (3.2).

The massive Poincaré generators act on ${}^m|j_1\dots j_s\rangle_t$ in exactly the same way as the massless generators on $|j_1\dots j_s\rangle_t$ in (2.14a,d–f). In particular, the deformation preserves the Hamiltonian P_0 and the generators M_{kl} of rotations. The deformation of the spacelike momenta gives immediately

$$\widetilde{P}_k|j_1\dots j_s\rangle_\omega = \sigma(\omega)\cdot P_k|j_1\dots j_s\rangle_\omega \quad \Rightarrow \quad \widetilde{P}_k = P_k\cdot\Big(1 - \frac{m^2}{P_0^2}\Big)^{\frac{1}{2}}. \quad (3.3)$$

The mass-shell condition

$$\widetilde{P}_k^2 = P_0^2 - m^2$$

is trivially fulfilled by (3.3). For the boosts, one gets

$$\begin{aligned}\widetilde{M}_{0k}|j_1\dots j_s\rangle_\omega &= \sigma(\omega)^{-s-\frac{1}{2}}\Big(\partial_\omega\big(\sigma(\omega)^{s+\frac{3}{2}}|k\dots\rangle_\omega\big) + \omega\sigma(\omega)^{s-\frac{1}{2}}\textstyle\sum_a \delta_{kj_a}|\dots\times\rangle_\omega\Big)\\ &= \big((s+\frac{3}{2})\sigma'(\omega) + \sigma(\omega)\partial_\omega\big)|k\dots\rangle_\omega + \omega\sigma(\omega)^{-1}\textstyle\sum_a \delta_{kj_a}|\dots\times\rangle_\omega.\end{aligned}$$

Using (2.14a,b,e,f), this can be seen to be equivalent to

$$\widetilde{M}_{0k} = \Big(M_{0k} - \frac{1}{2P_0}(DP_k + P_kD)\cdot\frac{m^2}{P_0^2}\Big)\cdot\Big(1-\frac{m^2}{P_0^2}\Big)^{-\frac{1}{2}} \tag{3.4}$$

(where the operator ordering has been adjusted so as to match the coefficient $s+\frac{3}{2}$). Because the generators on the subspace $E_m\mathcal{H}_0$ arise by the unitary conjugation (3.2) of the generators on $\mathcal{H}_m$, they are self-adjoint and satisfy the Poincaré commutation relations. Indeed, the hermiticity, as well as the commutator $i[P_0,\widetilde{M}_{0k}] = -\widetilde{P}_k$, can be verified without much effort. The explicit verification of the commutation relation $i[\widetilde{M}_{0k},\widetilde{M}_{0l}] = M_{kl}$ requires another cumbersome computation, which gives

$$i[\widetilde{M}_{0k},\widetilde{M}_{0l}] = \Big(M_{kl} + \big(M_{0k}P_l - M_{0l}P_k\big)\frac{m^2}{P_0^3}\Big)\Big(1-\frac{m^2}{P_0^2}\Big)^{-1}.$$

This equals M_{kl} on $E_m\mathcal{H}_0$ because $M_{0k}P_l+M_{l0}P_k+M_{kl}P_0 = \varepsilon_{klj}W^j$, and the Pauli-Lubanski operator $W^\mu = \frac{1}{2}\varepsilon^{\mu\nu\kappa\lambda}M_{\nu\kappa}P_\lambda$ vanishes in the massless scalar representation.

4 Summary

We have given two families of examples of spacelike deformations that allow to construct new quantum field theories by fixing the restriction of a given QFT to the time axis, and deforming only the "transverse" symmetry generators. The remarkable feature is that the scheme admits the change of discrete quantum numbers (the helicity in our first example). In both cases, it is true that we knew the expected deformation from the outset. But only in the mass deformation case did we use this knowledge to compute the deformed generators. In contrast, Prop. 2.2 is a uniqueness result, once the subspace is specified on which the deformation is supposed to be defined.

Both instances of spacelike deformation presented here make essential use of the envelopping algebra of the Lie algebra of the respective spacetime symmetry group (conformal, resp. Poincaré). It is a noticeable feature that in both cases, one extra element (the charge Q in the higher helicity case, the dilation operator D in the massive case) is needed for the deformation.

From the underlying pattern of inclusions of Hilbert spaces, we expect that one can deform any given helicity $h'\geq 0$ to a helicity $h>h'$, and any given mass $m\geq 0$ to a mass $m>m'$. On the other hand, increasing mass and spin simultaneously might not be possible by lack of an inclusion of one-particle representations of the subgroup fixing the time axis.

The case of interacting theories will need methods going beyond representation theory of spacetime symmetry groups.

5 Outlook

Our constructions may give insights into the modular theory of local algebras for massive theories [11], that is not as well known as for massless theories. Let us explain what we have in mind.

In a generic QFT, if the local algebras $A(O_I) = A(I)$ for doublecones O_I spanned by an interval I along the time axis are given, then they are defined for general doublecones by the adjoint action of Poincaré transformations. In Sect. 2.3 and Sect. 3, the deformed local algebras on the time axis arise just by restriction of the undeformed local algebras to the respective second-quantized subspace $\Gamma(E^{(h)})\mathcal{F}_0$ or $\Gamma(E_m)\mathcal{F}_0$.

In the case of helicity deformations, one may adopt a different point of view, referring only to the representations of the conformal group. Namely, given a unitary representation $\widetilde{U}$ of $\mathsf{M\ddot{o}b}$ on $\mathcal{H}$ and its extension by the anti-unitary PCT operator J, the Brunetti-Guido-Longo construction [4] (BGL) allows to define a real Hilbert space $H(\mathbb{R}_+) \subset \mathcal{H}$ such that $H(\mathbb{R}_+) \cap iH(\mathbb{R}_+) = \{0\}$ and $\overline{\mathbb{C}\,H(\mathbb{R}_+)} = \mathcal{H}$. This definition uses only the dilations and J. Acting with $\widetilde{U}(g)$, $g \in \mathsf{M\ddot{o}b}$, one obtains a net of real standard subspaces $I \mapsto H(I)$ on the intervals of the circle. This net of subspaces is local in the sense that the symplectic complement $H(I)' \equiv (iH(I))^\perp$ of $H(I)$ coincides with $H(I')$, where I' is the complement of I in S^1 and orthogonality $\perp$ refers to the real part of the scalar product. Upon second quantization, these properties turn into locality of a Möbius covariant chiral net of local algebras with the Reeh-Schlieder property. It trivially restricts to a net on the time axis by deleting the point $-1 \in S^1$ and identifying $S^1\backslash\{-1\}$ with $\mathbb{R}$ via the Cayley transform.

If the unitary representation of $\mathsf{M\ddot{o}b}$ extends to a representation of the four-dimensional conformal group on $\mathcal{H}$, then the net of standard subspaces on S^1 extends to a conformally covariant net $O \mapsto H(O)$ on the four-dimensional Dirac manifold, that in turn restricts to a net on Minkowski spacetime. By second quantization, one obtains a Huygens local net of local algebras $O \mapsto A(O)$. "Huygens locality" (= commutativity also at timelike distance) is a consequence of the locality along the time axis, that is guaranteed by the BGL construction.

By construction, the modular group of $H(\mathbb{R}_+)$ is given by the dilations, and that of $H(I)$ is the one-parameter subgroup of $\mathsf{M\ddot{o}b}$ that fixes the interval I. It follows that the modular groups of the local algebras $A(O)$ (O a doublecone or a wedge) in the vacuum state are the subgroups of the conformal group (conjugate to boost subgroups) that fix the doublecone or wedge O. In the construction of Sect. 2.3, the one-particle space is given by $E^{(h)}\mathcal{H}$, the representation of the Möbius group remains undeformed, and the local subspaces and local algebras away from the time axis are constructed with the deformed translations and boosts. Because the projection $E^{(h)}$ commutes with the representation of $\mathsf{M\ddot{o}b}$, it is automatic that the modular groups on the time axis coincide with those of the scalar field restricted to $E^{(h)}\mathcal{H}$, and away from the time axis are conjugate by deformed Poincaré transformations.

The situation is very different in the mass deformation of Sect. 3. Because the spectral projection E_m does not commute with the dilations, the latter are not defined on the subspace $\mathcal{H}_m$, and the BGL construction is not possible. Indeed, it is well-known that in the massive case, $H_m(\mathbb{R}_+)$ (to be identified with $H_m(V_+)$ in the net on Minkowski spacetime) has trivial symplectic complement ([9, 10]), in contrast to the duality $(iH_0(\mathbb{R}_+))^\perp = H_0(\mathbb{R}_-)$ in the massless case. On the other hand, we know that the massive local subspace $H_m(I)$ of an interval I on the time axis coincides with the local subspace $H_m(O_I)$ for the doublecone O_I spanned by I; and by the work [5] of Eckmann and Fröhlich, we have a local unitary equivalence between the massive and massless time-zero algebras. Specifically, there is a unitary operator U_R such that for intervals $I_r = (-r, r) \subset \mathbb{R}$ symmetric around $t = 0$ and $r < R$, one has $H_m(O_r) = U_R H_0(O_r)$ where O_r is the causal completion of the time-zero ball of radius r. Thus, the modular groups of $H_m(O_r)$ are, for $r < R$, conjugate to the known modular groups of $H_0(O_r)$ by U_R. Increasing R, the unitary U_R will change, but the subspaces $H(I_r)$ for $r < R$ and their modular groups remain unchanged. Thus, the modular groups for $r < R_1 < R_2$ commute with $U_{R_2}U^*_{R_1}$, and a more detailed investigation of the unitaries U_R would be worthwhile to get a first insight into the hitherto unknown massive modular groups.

This information about the modular groups then passes to arbitary doublecones via the adjoint action of the deformed translations and boosts, as constructed in Sect. 3.

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