# RENORMALIZATION OF DETERMINANT LINES IN QUANTUM FIELD THEORY.

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ABSTRACT. On a compact manifold $M$, we consider the affine space $\mathcal{A}$ of non-self-adjoint perturbations of some invertible elliptic operator acting on sections of some Hermitian bundle, by some differential operator of lower order.

We construct and classify all complex analytic functions on the Fréchet space $\mathcal{A}$ vanishing exactly over non-invertible elements, having minimal growth at infinity along complex rays in $\mathcal{A}$ and which are obtained by local renormalization, a concept coming from quantum field theory, called *renormalized determinants*. The additive group of local polynomial functionals of finite degrees acts freely and transitively on the space of renormalized determinants. We provide different representations of the renormalized determinants in terms of spectral zeta determinants, Gaussian Free Fields, infinite product and renormalized Feynman amplitudes in perturbation theory in position space à la Epstein–Glaser.

Specializing to the case of Dirac operators coupled to vector potentials and reformulating our results in terms of determinant line bundles, we prove our renormalized determinants define some complex analytic trivializations of some holomorphic line bundle over $\mathcal{A}$. This relates our results to a conjectural picture from some unpublished notes by Quillen [61] from April 1989.

## 1. INTRODUCTION.

Let $(M, g)$ be a smooth, closed, compact Riemannian manifold. The aim of the present paper is to study the analytical properties and the renormalization of a class of functional determinants defined on some affine space $\mathcal{A}$ of non self-adjoint operators that we divide in two classes:

- in the first class, we consider perturbations of the form $\Delta + V$ of some given invertible, self-adjoint, generalized Laplacian $\Delta$ acting on some fixed Hermitian bundle $E \mapsto M$ where $V \in C^\infty(End(E))$ is a smooth potential,
- in the second class, we look at perturbations of the form $D + A$ of some invertible twisted Dirac operator $D : C^\infty(E_+) \mapsto C^\infty(E_-)$ acting between Hermitian bundles $E_\pm$ by some term $A \in C^\infty(Hom(E_+, E_-))$. By invertible, we mean the Fredholm index of $D$ equals 0 and $\ker(D) = \{0\}$.

We consider lower order perturbations since $A$ and $V$ are local operators of order 0.

1.0.1. *Quantum field theory interacting with some external potential.* Let us briefly give the physical motivations underlying our results which are stated in purely mathematical terms. The reader uninterested by the physics can safely skip this part. Inspired by recent works in mathematical physics [19, 20, 21, 22, 31, 30] and classical works of Schwinger [67] [45, Chapter

4 p. 163], our original purpose is to understand the problem of renormalization of some Euclidean quantum field $\phi$ defined on $M$ interacting with a classical external field which is not quantized [1]. For instance, consider the Laplace–Beltrami operator $\Delta : C^\infty(M) \mapsto C^\infty(M)$ defined from the metric $g$ on $M$, corresponding to the Dirichlet action functional:

$$S(\phi) = \int_M (\phi\Delta\phi)\, dv \tag{1.1}$$

and their perturbations by some external potential $V \in C^\infty(M, \mathbb{R}_{\geqslant 0})$ which corresponds to the perturbed Dirichlet action functional

$$S(\phi) = \int_M \left(\phi\Delta\phi + V\phi^2\right) dv. \tag{1.2}$$

A classical problem in quantum field theory is to define the *partition function* of a theory, usually represented by some ill–defined functional integral. In the bosonic case, it reads:

$$Z\,(V) = \int [d\phi] \exp\left(-\frac{1}{2}\int_M \left(\phi\Delta\phi + V\phi^2\right) dv\right) \tag{1.3}$$

where $V \in C^\infty(M, \mathbb{R}_{>0})$ plays the role of a position dependent mass which is viewed as an external field coupled to the Gaussian Free Field $\phi$ to be quantized. The external field can also be the metric $g$ [2] in the study of *gravitational anomalies* in the physics litterature or *gauge fields*, which is the physical terminology for connection 1–forms, in the study of *chiral anomalies*.

In fact, according to Stora [75, 56], the physics of chiral anomalies [27, 56, 2] can be understood in the case where we have a quantized fermion field interacting with some *gauge field* which is treated as an external field. Consider the quadratic Lagrangian $\Psi_- D_A \Psi_+$ where $\Psi_\pm$ are *chiral* fermions, $D_A$ is a twisted half–Dirac operator acting from sections of positive spinors $S_+$, to negative spinors $S_-$, see example 1 below for a precise definition of $D_A$. In this case, the corresponding ill–defined functional integral reads

$$Z\,(A) = \int [\mathcal{D}\Psi_- \mathcal{D}\Psi_+] \exp\left(\int_M \langle \Psi_-, D_A\Psi_+\rangle\right)$$

where we are interested on the dependence in the gauge field $A$.

1.0.2. *Functional determinants in geometric analysis.* The above two problems can be formulated as the mathematical problem of constructing functional determinants $V \mapsto \det(\Delta + V)$ and $A \mapsto \det(D^*(D + A))$ with nice functional properties where we are interested in the dependence in the external potentials $V$ and $A$. In global analysis, functional determinants also appear in the study of the analytic torsion by Ray–Singer [63] and more generally as metrics of determinant line bundles as initiated by Quillen [62, 4] where he considered some affine space $\mathcal{A}$ of Cauchy-Riemann operators $D+\omega$ acting on some fixed vector bundle $E \mapsto X$ over a compact Riemann surface $X$ where $D$ is fixed and the perturbation $\omega$ lives in the linear space of $(0,1)$-forms on $X$ with values in the bundle $End(E)$. The metrics on $X$ and in $E$ induce a metric in the determinant (holomorphic line) bundle $\det(\mathrm{Ind}(D)) = \Lambda^{top}\ker(D)^* \otimes \Lambda^{top}\mathrm{coker}(D)$ over $\mathcal{A}$. As Quillen showed in [62], if this metric in the bundle $\det(\mathrm{Ind}(D))$ is divided by the

[1] sometimes called *background field* in the physical litterature

function $\det_\zeta(D^*D)$ (here $\det_\zeta$ is the zeta regularized determinant of the Laplacian $D^*D$), then the canonical curvature form of this metric, the first Chern form, coincides with the symplectic form of the natural Kähler metric on $\mathcal{A}$. An important consequence of the above observation, stated as a [62, Corollary p. 33], is that if one multiplies the Hermitian metric on $\det(\mathrm{Ind}(D))$ by $e^q$ where $q$ is the natural Kähler metric on $\mathcal{A}$, then the corresponding Chern connection is **flat**. From the contractibility of $\mathcal{A}$, one deduces the existence of a **global holomorphic trivialization** of $\det(\mathrm{Ind}(D)) \mapsto \mathcal{A}$ and the image of the canonical section of $\det(\mathrm{Ind}(D))$ by this trivialization is an analytic function on $\mathcal{A}$ vanishing over non–invertible elements.

Building on some ideas from the work of Perrot [56, 58, 57] and some unpublished notes from Quillen's notebook [61] [2], we attempt to relate the problem of constructing renormalized determinants with the construction of holomorphic trivializations of determinant line bundles over some affine space $\mathcal{A}$ of perturbations of some fixed operator by some differential operator of lower order which plays the role of the external potential.

1.0.3. *Quillen's conjectural picture.* In some notes on the 30th of April 1989 [61, p. 282], with the motivation to make sense of the technique of adding local counterterms to the Lagrangian used in renormalized perturbation theory, Quillen proposed to give an interpretation of QFT partition functions in terms of determinant line bundles over the space of Dirac operator coupled to a gauge potential drawing a direct connection between the two subjects. The approach he outlined insists on constructing *complex analytic trivializations* of the determinant line bundle without mentioning any construction of Hermitian metrics on the line bundle which seems different from the original approach he pioneered [62] and the Bismut–Freed [4] definition of determinant line bundle for families of Dirac operators.

To explain this connection, we recall that for a pair $\mathcal{H}_0, \mathcal{H}_1$ of complex Hilbert spaces, there is a **canonical holomorphic line bundle Det** $\longmapsto \mathrm{Fred}_0(\mathcal{H}_0, \mathcal{H}_1)$ where $\mathrm{Fred}_0(\mathcal{H}_0, \mathcal{H}_1)$ is the space of Fredholm operators of **index** 0 with fiber $\mathbf{Det}_B \simeq \Lambda^{top}\ker(B)^* \otimes \Lambda^{top}\mathrm{coker}(B)$ and canonical section $\sigma$ [62, p. 32] [70, p. 137–138]. Consider the **complex affine space** $\mathcal{A} = D + C^\infty(Hom(E_+, E_-))$ of perturbations of some fixed invertible Dirac operator $D$ by some **differential operator** $A \in C^\infty(Hom(E_+, E_-))$ of order 0. We denote by $L^2(E_+)$ the space of $L^2$ sections of $E_+$. Then the map $\iota : D + A \in \mathcal{A} \mapsto Id + D^{-1}A \in \mathrm{Fred}_0\left(L^2(E_+), L^2(E_+)\right)$ allows to pull–back the holomorphic line bundle **Det** as a holomorphic line bundle $\mathcal{L} = \iota^*\mathbf{Det} \mapsto \mathcal{A}$ over the affine space $\mathcal{A}$ with canonical section $\underline{\det} = \iota^*\sigma$. We insist that we view $C^\infty(Hom(E_+, E_-))$ as a $\mathbb{C}$–vector space, elements in $C^\infty(Hom(E_+, E_-))$ need not preserve Hermitian structures. According to Quillen [61, p. 282], the relation with QFT goes as follows, *one gives a meaning to the functional integrals*

$$A \quad \mapsto \quad \int \mathcal{D}\Psi_+ \mathcal{D}\Psi_- e^{\int_M \langle \Psi_-, D_A \Psi_+ \rangle}, \tag{1.4}$$

[2] made available by the Clay foundation at http://www.claymath.org/library/Quillen/Working_papers/quillen1989/1989-2.pdf

[3] *by trivializing the determinant line* . In other words, denoting by $\mathcal{O}(\mathcal{L})$ (resp $\mathcal{O}(\mathcal{A})$) the holomorphic sections (resp functions) of $\mathcal{L}$ (resp on $\mathcal{A}$), we aim at constructing a **holomorphic trivialization** of the line bundle $\tau : \mathcal{O}(\mathcal{L}) \longmapsto \mathcal{O}(\mathcal{A})$ so that the **image** $\tau(\underline{\det})$ of the canonical section $\underline{\det}$ by this trivialization is an entire function $f(P+\mathcal{V})$ on $\mathcal{A}$ vanishing exactly over the set $Z$ of non-invertible elements of $\mathcal{A}$. In some sense, this should generalize the original construction of Quillen of the holomorphic trivialization of the determinant line bundle over the space of Cauchy–Riemann operators [62]. Furthermore, Quillen [61, p. 284] writes:

**These considerations lead to the following conjectural picture. Over the space $\mathcal{A}$ of gauge fields there should be a principal bundle for the additive group of polynomial functions of degree $\leqslant d$ where $d$ bounds the trace which have to be regularized. The idea is that near each $A \in \mathcal{A}$ we should have a well–defined trivialization of $\mathcal{L}$ up to $\exp$ of such a polynomial. Moreover, we should have a flat connection on this bundle.**

To address this conjectural picture, we follow a backward path compared to [62]. Instead of constructing some Hermitian metric then a flat connection on $\mathcal{L} \mapsto \mathcal{A}$ to trivialize the bundle, we prove in Theorem 3 an infinite dimensional analog of the classical Hadamard factorization Theorem 1 in complex analysis. We classify all determinant-like functions such that:

- They are entire functions on $\mathcal{A}$ with minimal growth at infinity, a concept with is defined below as the *order* of the entire function, vanishing over non-invertible elements in $\mathcal{A}$.
- Their differentials should satisfy some simple identities reminiscent of the situation for the usual determinant in finite dimension.
- They are obtained from a renormalization by subtraction of local counterterms, a concept coming from quantum field theory which will be explained below in paragraph 2.6.

Trivializations of $\mathcal{L}$ are simply obtained by dividing the canonical section of $\mathcal{L}$ by the constructed determinant-like functions as showed in Theorem 4. A nice consequence of our investigation is a new factorization formula for zeta regularized determinant (3.1),(3.2) in terms of Gohberg–Krein's regularized determinants. We show that our renormalized determinants are not canonical and there are some ambiguities involved in their construction of the form $\exp(P)$ where $P$ is a local polynomial functional of $A$. Then we show that the additive group of local polynomial functionals of $A$, sometimes called the renormalization group of Stüeckelberg–Petermann in the physics litterature, acts freely and transitively on the space of renormalized determinants we construct.

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[3] $D_A$ is the Dirac operator coupled to the gauge potential $A$ as described in [73, section 3 p. 325]

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1.1. **Notations.** $dv$ is used for a smooth density $|\Lambda^{top}|M$ on $M$. In the sequel, for every pair $(B_1, B_2)$ of Banach spaces, $\mathcal{B}(B_1, B_2)$ denotes the Banach space of bounded operators from $B_1 \mapsto B_2$ endowed with the norm $\|.\|_{\mathcal{B}(B_1,B_2)}$. For any vector bundle $E$ on $M$, we denote by $\Psi^{\bullet}(M, E)$ the algebra of pseudodifferential operators on the manifold $M$ acting on sections of the bundle $E$ and when there is no ambiguity we will sometimes use the short notation $\Psi(M)$. $C^k(E), \|.\|_{C^k(E)}, k \in \mathbb{N}$ denotes the Banach space of continuous sections of $E$ of regularity $C^k$, $H^s(E), s \in \mathbb{R}$ denotes Sobolev sections of $E$ endowed with the norm $\|.\|_{H^s(E)}$ that we shall sometimes write $H^s, \|.\|_{H^s}$ for simplicity[4] and finally $L^p(E), 1 \leqslant p \leqslant +\infty$ denotes $L^p$ sections of $E$ endowed with the norm $\|.\|_{L^p(E)}$. $C^\infty_c(U, E)$ are smooth sections of $E$ compactly supported on $U$.

For any pair $(E, F)$ of bundles over $M$, for $C^m$ Schwartz kernels $K$ of operators from $C^m(E) \mapsto C^m(F)$ which are elements of $C^m(M \times M, F \boxtimes E^*)$, we denote by $\|K\|_{C^m(M\times M)}$ their $C^m$ norm which is not to be confused with the operator norm $\|K\|_{\mathcal{B}(C^m(E),C^m(F))}$.

For any Hilbert space $H$, we denote by $\mathcal{I}_p \subset \mathcal{B}(H, H)$ the Schatten ideal of compact operators whose $p$-th power is trace class endowed with the norm $\|.\|_p$ defined as $\|A\|_p = \sum_{\lambda\in\sigma(A)} |\lambda|^p$ where the sum runs over the singular values of $A$.

## 2. Preliminary material.

The goal of this section is to introduce enough material to state precisely our main results. We begin with some classical results on entire functions on $\mathbb{C}$ with given zeros. Since we view functional determinants as infinite dimensional analogues of entire functions with given zeros, we need to recall classical results on holomorphic functions in Fréchet spaces. We conclude the introductory part by discussing Fredholm determinants and their generalizations by Gohberg–Krein which are also viewed as entire functions with given zeros on some infinite dimensional Banach or Fréchet spaces. This also serves as motivation for our main results.

2.1. **Entire functions with given zeros on $\mathbb{C}$.** In this paragraph, we recall some classical results on entire functions with given zeros. The **order** $\rho(f) \geqslant 0$ of an entire function $f$ is the infimum of all the real numbers $\rho$ such that for some $A, K > 0$, for all $z \in \mathbb{C}$ $|f(z)| \leqslant Ae^{K|z|^\rho}$. The **critical exponents** of a sequence $|a_n| \to +\infty$, is the infimum of all $\alpha > 0$ such that $\sum_n \frac{1}{|a_n|^\alpha} < +\infty$. Finally the **genus** of $f$ is the order of vanishing of $f$ at $z = 0$. The **divisor** of an entire function $f$ is the set of zeros of $f$ counted with multiplicity. We recall a classical Theorem due to Hadamard on the structure of entire functions with given zeros [65, p. 78–81], [74, Thm 5.1 p. 147] (see also [54, p. 60]):

[4] when $s < 0$ these are distributional sections

**Theorem 1** (Hadamard's factorization Theorem)**.** *Let $(a_n)_{n\in\mathbb{N}}$ be some sequence such that $\sum_n |a_n|^{-(p+1)} < +\infty$ but $\sum_n |a_n|^{-p} = \infty$. Then any entire function $f$ whose divisor $Z(f) = \{a_n | n \in \mathbb{N}\}$ has order $\rho(f) \geqslant p$, and any entire function s.t. $Z(f) = \{a_n | n \in \mathbb{N}\}$ and $\rho(f) = p$ has a* ***representation*** *as:*

$$\boxed{f(z) = z^m e^{P(z)} \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right)} \tag{2.1}$$

*where $P$ is a polynomial of degree $p$, $E_p(z) = (1-z)e^{z+\frac{z^2}{2}+\dots+\frac{z^p}{p}}$ is a Weierstrass factor of order $p$ and $m$ is the genus of $f$.*

The lower bound on the order of $f$ follows from Jensen's formula. Observe that the entire functions produced by the Hadamard factorization Theorem are not unique due to the polynomial ambiguity which brings a factor $e^P$. We will meet this ambiguity again in Theorem 3 which is responsible for the renormalization group.

## 2.2. **Entire functions on a complex Fréchet space.**

2.2.1. *Smooth functions on Fréchet spaces.* In the present paper, we always work with Fréchet spaces of smooth sections of finite rank vector bundles over some compact manifold $M$. Smooth functions on Fréchet spaces will be understood in the sense of Bastiani [7, Def II.12], as popularized by Hamilton [40] in the context of Fréchet spaces and Milnor [76]. This means smooth functions are infinitely differentiable in the sense of Gâteaux and all the differentials $D^nF : U \times E^n \mapsto \mathbb{C}$ are jointly continuous on $U \times E^n$ [7, Def II.11]. We recall the notion of Gâteaux differentials and the correspondance between multilinear maps and distributional kernels since these will play a central role in our approach:

**Definition 2.1** (Gâteaux differentials and Schwartz kernels of multilinear maps)**.** *Let $B \mapsto M$ be some Hermitian vector bundle of finite rank on some smooth closed compact manifold $M$. For a smooth function $f : V \in C^\infty(M, B) \mapsto f(V) \in \mathbb{C}$ where $C^\infty(M, B)$ is the Fréchet space of smooth sections, the $n$-th differential*

$$D^n f(V, h_1, \dots, h_n) = \prod_{i=1}^{n} \frac{d}{dt_i} f(V + t_1h_1 + \dots + t_nh_n)|_{t_1=\dots=t_n=0} \tag{2.2}$$

*is multilinear continuous in $(h_1, \dots, h_n)$, hence it can be identified by the multilinear Schwartz kernel Theorem* [7, lemm III.6] *with the unique distribution $[\mathbf{D^nf(V)}]$ in $\mathcal{D}'(M^n, B^{\boxtimes n})$, called* ***Schwartz kernel of*** *$D^n f(V)$, s.t.*

$$\langle [\mathbf{D^nf(V)}], h_1 \boxtimes \dots \boxtimes h_n\rangle = D^n f(V, h_1, \dots, h_n) \tag{2.3}$$

*is jointly continuous in $(V; h_1, \dots, h_n) \in C^\infty(M, B)^{n+1}$* [7, Thm III.10].

In the sequel, to stress the difference beetween the $n$-th differential $D^n f(V)$ of a function $f$ at $V$ from its Schwartz kernel, we use the notation $[\mathbf{D^nf(V)}]$ for the Schwartz kernel. In the physics litterature, Gâteaux differentials of smooth functions on spaces of functions are often called **functional derivatives**. These functional derivatives play an important role in classical and quantum field theory and are usually represented (in fact identified) by their Schwartz kernels.

2.2.2. *Holomorphic functions on Fréchet spaces.* First, let us define what we mean by an entire function on a Fréchet space.

**Definition 2.2** (Holomorphic and entire functions on Fréchet spaces)**.** *Let $\Omega \subset E$ be some open subset in a Fréchet space $E$. A function $F : \Omega \subset E \mapsto \mathbb{C}$ is holomorphic if it is **smooth** and for every $V_0 \in \Omega$, the Taylor series of $F$ converges in some neighborhood of $V_0$* [5] *and $F$ coincides with its Taylor series:*

$$F(V_0 + h) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n F(V_0, h, \dots, h)$$

*where the r.h.s. converges absolutely. In case $\Omega = E$, $F$ will be called **entire**.*

## 2.3. **Fredholm and Gohberg–Krein's determinants as entire functions vanishing over non-invertible elements.**

2.3.1. *Fredholm determinants.* We briefly recall the definition of Fredholm determinant $\det_F(Id+B)$ for a trace class operator $B : H \mapsto H$ acting on some separable Hilbert space $H$ and relate them with functional traces of powers of $B$. These identities will imply that $\det_F$ is an example of entire function on the infinite dimensional space $(Id + \text{trace class})$ whose zeros are exactly the non-invertible operators.

**Definition 2.3** (Fredhom determinants)**.** *The Fredholm determinant* $\det_F(Id+B)$ *is defined in* [71, equation (3.2) p. 32] *as*

$$det_F(Id + B) = \sum_{k=0}^{\infty} Tr(\Lambda^k B) \tag{2.4}$$

*where $\Lambda^k B : \Lambda^k H \mapsto \Lambda^k H$ acting on the $k$-th exterior power $\Lambda^k H$ is trace class. Using the bound $\|\Lambda^k B\|_1 \leqslant \frac{\|B\|_1}{k!}$* [71, Lemma 3.3 p. 33]*, it is immediate that* $\det_F(Id+zB)$ *is an **entire** function in $z \in \mathbb{C}$ (see also* [36, Thm 2.1 p. 26]*).*

For any compact operator $B$, we will denote by $(\lambda_k(B))_k$ its eigenvalues counted with multiplicity. By [71, Theorem 3.7], the Fredholm determinant can be identified with a Hadamard product and is related to the functional traces by the following sequence of identities:

$$\det_F (Id + zB) = \prod_k (1 + z\lambda_k(B)) = \underbrace{\exp\left( \sum_{m=1}^{\infty} (-1)^{m+1} z^m Tr_{L^2}(B^m) \right)} \tag{2.5}$$

where the term underbraced involving traces is well–defined only when $|z|\|B\|_1 < 1$. Note the important fact that $\exp\left(\sum_{m=1}^{\infty}(-1)^{m+1} z^m Tr_{L^2}(B^m)\right)$ which is defined on the disc $\mathbb{D} = \{|z|\|B\|_1 < 1\}$ has analytic continuation as an entire function of $z \in \mathbb{C}$ and $B \mapsto \det_F(Id + B)$ is an entire function vanishing when $Id + B$ is non-invertible.

[5] for the Fréchet topology

2.3.2. *Gohberg–Krein's determinants.* Set $p \in \mathbb{N}$ and let $A$ belong to the Schatten ideal $\mathcal{I}_p \subset \mathcal{B}(H,H)$. Following [71, chapter 9], we consider the operator

$$R_p(A) = [(Id + A)\exp(\sum_{n=1}^{p-1} \frac{(-1)^n}{n} A^n) - I] \in \mathcal{I}_1$$

which is trace class by [71, Lemma 9.1 p. 75] since $A \in \mathcal{I}_p$. Then following [71, p. 75]:

**Definition 2.4** (Gohberg–Krein's determinants)**.** *For any integer $p \geqslant 2$, we define the Gohberg–Krein determinant $\det_p : Id + \mathcal{I}_p \subset \mathcal{B}(H,H) \mapsto \mathbb{C}$ as:*

$$det_p(Id + A) = det_F\left(Id + R_p(A)\right) \tag{2.6}$$

*where $\det_F$ is the Fredholm determinant. The quantity $\det_p$ is well defined since $B = R_p(A)$ is trace class.*

**Proposition 2.5.** *Both $\det_F : Id + \mathcal{I}_1 \mapsto \mathbb{C}$ and $\det_p : Id + \mathcal{I}_p \mapsto \mathbb{C}$ are entire functions vanishing exactly over non-invertible elements in the following sense:*

$$det_p\left(Id + B\right) = 0 \Leftrightarrow det_p\left(Id + zB\right) = (z-1)^{\dim(\ker(Id+B))}\left(C + \mathcal{O}(z-1)\right), C \neq 0.$$
$$det_F\left(Id + B\right) = 0 \Leftrightarrow det_F\left(Id + zB\right) = (z-1)^{\dim(\ker(Id+B))}\left(C + \mathcal{O}(z-1)\right), C \neq 0.$$

2.4. **Geometric setting.** In the present paragraph, we fix once and for all the assumptions and the general geometric framework of the main Theorems (2),(3),(4) and that we shall use in the sequel. For $E \mapsto M$ some smooth Hermitian vector bundle over the compact manifold $M$, we denote by $C^\infty(E)$ smooth sections of $E$. An operator $\Delta : C^\infty(E) \mapsto C^\infty(E)$ is called generalized Laplacian if the principal part of $\Delta$ is positive definite, symmetric (i.e. formally self-adjoint) and diagonal with symbol $g_{\mu\nu}(x)\xi^\mu\xi^\nu \otimes Id_{E_x}$ in local coordinates at $(x;\xi) \in T^*M$ where $g$ is the Riemannian metric on $M$. We are interested in the following two geometric situations:

**Definition 2.6** (Bosonic case)**.** *Let $(M,g)$ be a smooth, closed, compact Riemannian manifold and $E$ some Hermitian bundle on $M$. We consider the complex affine space $\mathcal{A}$ of perturbations of the form $\Delta + V$ where $V$ is a smooth endomorphism $V \in C^\infty\left(End(E)\right)$, and $\Delta : C^\infty(E) \mapsto C^\infty(E)$ is an invertible generalized Laplacian. The element $V \in C^\infty(End(E))$ is treated as external potential.*

**Definition 2.7** (Fermionic case)**.** *Let $(M,g)$ be a smooth, closed, compact Riemannian manifold. Slightly generalizing the framework described in* [73, section 3 p. 325–327] *in the spirit of* [3, def 3.36 p. 116]*, we are given some pair of isomorphic Hermitian vector bundles $(E_+, E_-)$ of finite rank over $M$ and an invertible, elliptic first order differential operator $D : C^\infty(E_+) \mapsto C^\infty(E_-)$ such that both $DD^* : C^\infty(E_-) \mapsto C^\infty(E_-)$ and $D^*D : C^\infty(E_+) \mapsto C^\infty(E_+)$ are generalized Laplacians where $D^*$ is the adjoint of $D$ induced by the metric $g$ on $M$ and the Hermitian metrics on the bundles $(E_+, E_-)$. We consider the complex affine space $\mathcal{A}$ of perturbations $D + A : C^\infty(E_+) \mapsto C^\infty(E_-)$ where $A \in C^\infty(Hom(E_+, E_-))$.*

Recall that in both cases, we perturb some fixed operator by a **local** operator of order 0. We next give an important example from the litterature which fits exactly in the fermionic situation:

**Example 1** (Quantized Spinor fields interacting with gauge fields)**.** *Assume $(M,g)$ is* ***spin of even dimension*** *whose scalar curvature is nonnegative and positive at some point on $M$. For example $M=\mathbb{S}^{2n}$ with metric $g$ close to the round metric. Then it is well–known that the complex spinor bundle $S\mapsto M$ splits as a direct sum $S=S_+\oplus S_-$ of isomorphic hermitian vector bundles, the classical Dirac operator $D:C^\infty(S)\mapsto C^\infty(S)$ is a formally self-adjoint, elliptic operator of Fredholm index 0 which is invertible by the positivity of the scalar curvature thanks to the Lichnerowicz formula* [50, Cor 8.9 p. 160].

*Consider an external hermitian bundle $\mathcal{F}\mapsto M$ which is coupled to $S$ by tensoring $(S_+\oplus S_-)\otimes\mathcal{F}=E_+\oplus E_-$. For any Hermitian connection $\nabla^{\mathcal{F}}$ on $\mathcal{F}$, we define the twisted Dirac operator $D_{\mathcal{F}}:C^\infty(S\otimes\mathcal{F})\mapsto C^\infty(S\otimes\mathcal{F})$, which is a first order differential operator of degree 1 w.r.t. the $\mathbb{Z}_2$ grading, $D_{\mathcal{F}}=c(e_i)\left(\nabla^S_{e_i}\otimes Id+Id\otimes\nabla^{\mathcal{F}}_{e_i}\right)$ near $x\in M$ where $(e_i)_i$ is a local orthonormal frame of $TM$ near $x$, $c(e_i)$ is the Clifford action of the local orthonormal frame $(e_i)_i$ of $TM$ on $S$. In the study of chiral anomalies, one is interested by the half–Dirac operator $D:C^\infty(S_+\otimes\mathcal{F})\mapsto C^\infty(S_-\otimes\mathcal{F})$. If $(M,g)$ has* ***positive scalar curvature*** *and the curvature of $\nabla^{\mathcal{F}}$ is small enough then $\dim\ker(D)=0$ and $Ind(D)=0$* [50, prop 6.4 p. 315]*. Two connections on $\mathcal{F}$ differ by an element $\mathfrak{A}\in\Omega^1(M,End(\mathcal{F}))$. So we may define perturbations $D+A$ of our half–Dirac operator $D$, induced by perturbations of $\nabla^{\mathcal{F}}$, of the form*

$$\boxed{D+A=c(e_i)\left(\nabla^S_{e_i}\otimes Id+Id\otimes\left(\nabla^{\mathcal{F}}_{e_i}+\mathfrak{A}(e_i)\right)\right).} \tag{2.7}$$

In the sequel, for a pair $(E,F)$ of bundles over $M$, we always identify an element $\mathcal{V}\in C^\infty(Hom(E,F))$, which is a $C^\infty$ section of the bundle $Hom(E,F)$ with the **corresponding linear operator** $\mathcal{V}:C^\infty(E)\mapsto C^\infty(F)$, in the scalar case this boils down to identifying a function $V\in C^\infty(M)$ with the multiplication operator $\varphi\in C^\infty(M)\mapsto V\varphi\in C^\infty(M)$. To avoid repetitions and to stress the similarities between bosons and fermions, we will often denote in the sequel (for problem 3.2, Theorems 3 and 4) $\mathcal{A}=P+C^\infty(Hom(E,F))$ for the affine space of perturbations of $P=\Delta$ of degree 2, $E=F$ in the bosonic case and of $P=D$ of degree 1, $E=E_+,F=E_-$ in the fermionic case.

| | Bosons | Fermions |
|---|---|---|
| Bundles $(E,F)$ | $(E,E)$ | $(E_+,E_-)$ |
| Principal part $P$ | Laplace $\Delta$ order 2 | chiral Dirac $D$ order 1 |
| Perturbation $\mathcal{V}$ | $V\in C^\infty(M,End(E))$ | $A\in C^\infty(M,Hom(E_+;E_-))$ |
| Affine space $\mathcal{A}$ | $\Delta+V$ | $D+A$ |

## 2.5. **Zeta regularization.**

2.5.1. *Defining complex powers by spectral cuts.* The usual method to construct functional determinants is the zeta regularization pioneered by Ray–Singer [63] in their seminal work on analytic torsion and relies on spectral or pseudodifferential methods [34, 69]. The reader should see also [55, 68] for some nice recent reviews of various methods to regularize traces and determinants. Let us review the definition, in our context, of such analytic regularization (see [6, section 3 p. 203] for a very nice summary of the main results on zeta determinants) keeping in mind the subtle point that we consider non-self-adjoint operators.

Let $M$ be a smooth, closed compact manifold and $E \mapsto M$ some Hermitian bundle. We denote by $\text{Diff}^1(M,E)$ the space of differential operators with $C^\infty$ coefficients of order 1 acting on $C^\infty(E)$. For every perturbation of the form $\Delta + B : C^\infty(E) \mapsto C^\infty(E)$ of an invertible, symmetric, generalized Laplacian $\Delta$ by some differential operator $B \in \text{Diff}^1(M,E)$, the operator $\Delta+B$ has a canonical closure from $H^2(M,E) \mapsto L^2(M,E)$ by ellipticity of $\Delta+B \in \Psi^2(M,E)$. In the notations from subsection 2.4, $B = V \in C^\infty(M, End(E))$ in the bosonic case has order 0 or $B = D^*A \in \text{Diff}^1(M,E_+)$ in the fermionic case in which case $B$ has order 1.

By the compactness of the resolvent $(\Delta + B - z)^{-1}$ and meromorphic Fredholm theory, $\Delta + B : H^2(M,E) \mapsto L^2(M,E)$ has discrete spectrum with finite multiplicity which we denote by $\sigma(\Delta + B) \subset \mathbb{C}$. Since the operator $\Delta + B$ is no longer self-adjoint, we must choose a **spectral cut** to define its complex powers. The operator $\Delta + B$ has principal angle $\pi$ since the value of the principal symbol $g_{\mu\nu}(x)\xi^\mu\xi^\nu$ of $\Delta + B$ never meets the ray $R_\pi = \{re^{i\pi}, r \geqslant 0\} = \mathbb{R}_{\leqslant 0}$. Furthermore, for $\Delta^{-1}B \in \mathcal{B}(L^2, L^2)$ small enough, the spectrum $\sigma(\Delta + B) \subset \mathbb{C}$ will not meet some conical neighborhood $\{re^{i\theta}, r \geqslant 0, \theta \in [\pi-\varepsilon, \pi+\varepsilon], \varepsilon > 0\}$ of $\mathbb{R}_{\leqslant 0}$, see Proposition 4.7 for more details. For such $B \in \text{Diff}^1(M)$, $\pi$ is an Agmon angle for $\Delta + B$ and $\Delta + B$ is said to be **admissible with spectral cut** $\pi$.

Since $\Delta + B$ is invertible, we choose some $\rho > 0$ s.t. the disc of radius $\rho$ does not meet $\sigma(\Delta + B)$, see Proposition 4.7. Then we define the contour [59, 10.1 p. 87–88] [48, p. 12]

$$\gamma = \{re^{i\pi}, \infty > r \geqslant \rho\} \cup \{\rho e^{i\theta}, \theta \in [\pi, -\pi]\} \cup \{re^{-i\pi}, \rho \leqslant r < \infty\}.$$

We define the complex powers as [6]:

$$(\Delta + B)_\pi^{-s} = \frac{i}{2\pi} \int_\gamma \lambda^{-s} (\Delta + B - \lambda)^{-1} d\lambda.$$

2.5.2. *The spectral zeta function and zeta determinants.* It is well known that the holomorphic family of operators $(\Delta + B)_\pi^{-s}$ is trace class for $Re(s) > \frac{\dim(M)}{2}$ and by the works of Seeley [34, 69], the spectral zeta function defined as

$$\zeta_{\Delta+B,\pi}(s) = Tr_{L^2}\left((\Delta + B)_\pi^{-s}\right) \tag{2.8}$$

has meromorphic continuation to the complex plane without poles at $s = 0$. In fact, much more general operators are considered in the work of Seeley who only requires ellipticity and the existence of an Agmon angle to define the spectral cut.

To formulate the spectral zeta function entirely in terms of the spectrum $\sigma(\Delta + B)$, note that $\sigma(\Delta + B) \cap \mathbb{R}_{\leqslant 0} = \emptyset$. Then using the classical branch of the logarithm on $\mathbb{C} \setminus \mathbb{R}_{\leqslant 0}$, we

[6] $\lambda^{-s} = e^{-s\log(\lambda)}$ where $\log(\lambda) = \log(|\lambda|) + i\arg(\lambda)$ for $-\pi \leqslant \arg(\lambda) \leqslant \pi$

can obtain an expression for the spectral zeta function as [48, Eq (2.14) p. 15]

$$\zeta_{\Delta+B,\pi}(s)=\sum_{\lambda\in\sigma(\Delta+B)}\lambda^{-s} \tag{2.9}$$

where the series on the r.h.s. converges absolutely for $Re(s)>\frac{\dim(M)}{2}$. This follows immediately from Lidskii's Theorem and the Weyl law for perturbations of self-adjoint, positive definite, elliptic operators [1, p. 238] due to Agranovich-Markus.

Let us comment on the above definition now for $B$ arbitrary in $\text{Diff}^1(M,E)$. When $B$ was small in the natural Fréchet topology of $\text{Diff}^1(M,E)$, it was unambiguous to define $\det_\zeta$ with the spectral cut at $\pi$ because we knew $\sigma(\Delta+B)\cap\mathbb{R}_{\leqslant 0}=\emptyset$. However if $B\in\text{Diff}^1(M,E)$ is chosen arbitrarily, $\sigma(\Delta+B)$ might intersect $\mathbb{R}_{\leqslant 0}$ and we may choose any other spectral cut in $(0,2\pi)$. In fact, the definition of complex powers and spectral zeta function may depend on the choice of spectral cut but the zeta determinant does not depend on the choice of angle $\theta$ provided $\theta\in(0,2\pi)$ since any such angle $\theta$ is a principal angle. This is due to the fact that we consider operators of the form $\Delta+B$ where $B$ has order 1 hence the leading symbol is self-adjoint of Laplace type. In fact, for any closed conical neighborhood of $\mathbb{R}_{\geqslant 0}$, only a finite number of eigenvalues of $\Delta+B$ lies outside this conical neighborhood as we discuss in Proposition 4.7. Said differently, for any angle $\theta\in(0,2\pi)$, there exists a conical neighborhood $L_{[\theta-\varepsilon,\theta+\varepsilon]}=\{z|\arg(z)\in[\theta-\varepsilon,\theta+\varepsilon]\}$ s.t. $L_{[\theta-\varepsilon,\theta+\varepsilon]}\cap\sigma(\Delta+B)$ is finite. So moving the cut in $(0,2\pi)$ only crosses a finite number of eigenvalues which implies by [48, Remark 2.1] [6, 3.10 p. 206] that $\det_\zeta(\Delta+B)$ does not depend on $\theta\in(0,2\pi)$. So this justifies why in the sequel we may write unambiguously $\det_\zeta(\Delta+B)$ where we choose any spectral cut $\theta\in(0,2\pi)$ to define $\det_\zeta$.

**Definition 2.8** (Spectral zeta determinant)**.** *The zeta determinant* $\det_\zeta$ *is defined as:*

$$det_\zeta(\Delta+B)=\exp\left(-\zeta'_{\Delta+B,\pi}(0)\right). \tag{2.10}$$

We next specialize our definitions of zeta determinants in the bosonic and fermionic cases:

**Definition 2.9** (Zeta determinants for bosons and fermions.)**.** *We use the geometric setting for bosons and fermions defined in paragraph 2.4. For bosons, we define the corresponding zeta determinant as a map*

$$V\in C^\infty(End(E))\mapsto det_\zeta(\Delta+V). \tag{2.11}$$

*For fermions, following* [73, p. 329]*, we define the corresponding zeta determinant as a map*

$$A\in C^\infty(Hom(E_+,E_-))\longmapsto det_\zeta\left(D^*(D+A)\right). \tag{2.12}$$

2.6. **Determinants renormalized by subtraction of local counterterms.** In order to give a precise definition of locality, we recall the definition of smooth local functionals.

**Definition 2.10** (Local polynomial functionals)**.** *A map* $P:V\in C^\infty(Hom(E,F))\mapsto P(V)\in\mathbb{C}$ *is called local polynomial functional if* $P$ *is smooth in the Fréchet sense and there exists* $k\in\mathbb{N}$, $\Lambda:V\in C^\infty(Hom(E,F))\longmapsto\Lambda(V)\in C^\infty(M)\otimes_{C^\infty(M)}|\Lambda^{top}|M$ *s.t. for all* $x\in M$, $\Lambda(V)(x)$ *depends polynomially on* $k$*-jets of* $V$ *at* $x$ *and* $P(V)=\int_M\Lambda(V)$*. The* *vector space of local polynomial functionals of degree $d$ depending on the $k$-jets is denoted by* $\mathcal{O}_{loc,d}\left(J^kHom(E,F)\right)$.

With the above notion of local functionals, we can explain the problem of renormalization of determinants by subtraction of local counterterms as follows. We denote by $\mathbb{C}[\varepsilon^{-\frac{1}{2}},\log(\varepsilon)]$ the ring of polynomials in $\log(\varepsilon)$ and inverse powers $\varepsilon^{-\frac{1}{2}}$. If we perturbed some elliptic operator $P$ by any smoothing operator $\mathcal{V}\in\Psi^{-\infty}$, then the Fredholm determinant

$$\mathcal{V}\in\Psi^{-\infty}\mapsto \det_F(Id+P^{-1}\mathcal{V})$$

would be a natural entire function on $\mathcal{A}=P+\Psi^{-\infty}$ vanishing over non-invertible elements. Unfortunately, the perturbations $\mathcal{V}\in C^\infty(M,Hom(E,F))$ in both bosonic and fermionic case, are viewed as pseudodifferential operators $\mathcal{V}\in\Psi^0(M)$ of degree 0 hence $\mathcal{V}\in\Psi^0(M)$ is surely not smoothing. Therefore, the Fredholm determinant $\det_F(Id+P^{-1}\mathcal{V})$ will be ill–defined since $Id+P^{-1}\mathcal{V}$ does not belong to the determinant class $Id+\mathcal{I}_1$. This is why we need to mollify the operator $\mathcal{V}$ by some family $(\mathcal{V}_\varepsilon)_\varepsilon$ of smoothing operators approximating $\mathcal{V}$ and consider the family $\det_F\left(Id+P^{-1}\mathcal{V}_\varepsilon\right)$ of Fredholm determinants which becomes potentially singular when $\varepsilon\to0^+$ and try to absorb the singularities created when $\varepsilon\to0^+$ by some multiplicative counterterm. This is formalized as follows:

**Definition 2.11** (Determinants renormalized by subtraction of local counterterms.). *If there is some family $(\mathcal{V}_\varepsilon)_\varepsilon$ of smoothing operators approximating $\mathcal{V}$, $\mathcal{V}_\varepsilon \underset{\varepsilon\to0^+}{\to} \mathcal{V}$ in $\Psi^{+0}(M)$, some family of local polynomial functionals $P_\varepsilon=\int_M\Lambda_\varepsilon(.)\in\mathcal{O}_{loc,d}\left(J^kHom(E,F)\right)\otimes_{\mathbb{C}}\mathbb{C}[\varepsilon^{-\frac{1}{2}},\log(\varepsilon)]$ called* ***local counterterms****, such that the limit*

$$\mathcal{V}\in C^\infty(M,Hom(E,F))\mapsto\lim_{\varepsilon\to0^+}\exp\left(-\int_M\Lambda_\varepsilon(\mathcal{V}(x))\right)det_F\left(Id+P^{-1}\mathcal{V}_\varepsilon\right)$$

*makes sense as entire function of $\mathcal{V}$ [7]. Then $\lim_{\varepsilon\to0^+}\exp\left(-\int_M\Lambda_\varepsilon(\mathcal{V}(x))\right)det_F\left(Id+P^{-1}\mathcal{V}_\varepsilon\right)$ is the* ***renormalization*** *of the singular family $det_F\left(Id+P^{-1}\mathcal{V}_\varepsilon\right)$* ***by subtraction of local counterterms****.*

## 3. Main Theorems.

3.1. **Main Theorem on the structure of zeta determinants.** Our first main result gives a factorization formula for zeta determinants in terms of Gohberg–Krein's determinants and renormalized Feynman amplitudes. In fact, the reader can think of this result as some infinite dimensional analog of Hadamard's factorization Theorem 1 in infinite dimension where we think of $\det_\zeta$ as an **entire function** on the affine space $\mathcal{A}$. In the sequel, we denote by $d_n\subset M^n$, the deepest diagonal $\{(x,\dots,x)\in M^n \text{ s.t. } x\in M\}\subset M^n$ and by $N^*\left(d_n\subset M^n\right)$ the conormal bundle of $d_n$. We use the notion of wave front set $WF(t)$ of a distribution $t$ to describe singularities of $t$ in cotangent space and refer to [43, chapter 8] for the precise definitions.

For $a\in\mathbb{R}$, we denote by $[a]=\sup_{k\in\mathbb{Z},k\leqslant a}k$. The bundle of densities on a manifold $X$ will be denoted by $|\Lambda^{top}|X$.

[7] $\det_F\left(Id+P^{-1}\mathcal{V}_\varepsilon\right)$ is well defined for $\varepsilon>0$ since $P^{-1}\mathcal{V}_\varepsilon\in\Psi^{-\infty}$

**Theorem 2.** *The zeta determinants from definition 2.9 are entire functions on $\mathcal{A}$ satisfying the factorization formula:*

$$\begin{aligned} det_\zeta\left(\Delta+V\right) &= e^{Q(V)}det_p\left(Id+\Delta^{-1}V\right), p=[\frac{d}{2}]+1 \text{ in bosonic case} \quad (3.1)\\ det_\zeta(D^*(D+A)) &= e^{Q(A)}det_p\left(Id+D^{-1}A\right), p=d+1 \text{ in fermionic case} \quad (3.2)\end{aligned}$$

*where* $\det_p$ *are Gohberg–Krein's determinants,* $Q(V)$ *(resp* $Q(A)$*) has degree* $[\frac{d}{2}]$ *(resp* $d$*)* [8].

*We furthermore have the following properties*

- *an exponential bound on the growth:*

$$\begin{aligned} \left|det_\zeta\left(\Delta+V\right)\right| &\leqslant Ce^{K\|V\|^{[\frac{d}{2}]+1}_{C^{d-3}}}\\ \left|det_\zeta(D^*(D+A))\right| &\leqslant Ce^{K\|A\|^{d+1}_{C^{d-1}}},\end{aligned}$$

- *an identity for all the Gâteaux differentials:*

$$\begin{aligned} \frac{(-1)^{n-1}}{n-1!}D^n\log det_\zeta\left(\Delta+V,V_1,\dots,V_n\right) &= Tr_{L^2}\left((\Delta+V)^{-1}V_1\dots(\Delta+V)^{-1}V_n\right),\\ \text{if } supp(V_1)\cap\dots\cap supp(V_n)=\emptyset &\\ \frac{(-1)^{n-1}}{n-1!}D^n\log det_\zeta\left(D^*(D+A),A_1,\dots,A_n\right) &= Tr_{L^2}\left((D+A)^{-1}A_1\dots(D+A)^{-1}A_n\right),\\ \text{if } supp(A_1)\cap\dots\cap supp(A_n)=\emptyset &\end{aligned}$$

- *a bound on the wave front set of the Schwartz kernels of all the Gâteaux differentials:*

$$\begin{aligned} WF\left([\mathbf{D}\log det_\zeta\left(\mathbf{\Delta}\right)]\right)=\emptyset,&\\ \forall n\geqslant 2, WF\left([\mathbf{D^n}\log det_\zeta\left(\mathbf{\Delta}\right)]\right)\cap T^*_{d_n}M^n &\subset N^*(d_n\subset M^n),\\ WF\left([\mathbf{D}\log det_\zeta\left(\mathbf{D^*D}\right)]\right)=\emptyset,&\\ \forall n\geqslant 2, WF\left([\mathbf{D^n}\log det_\zeta\left(\mathbf{D^*D}\right)]\right)\cap T^*_{d_n}M^n &\subset N^*(d_n\subset M^n),\end{aligned}$$

  *where* $[\mathbf{D^n}\log det_\zeta\left(\mathbf{\Delta}\right)]\in\mathcal{D}'(M^n)$ *(resp* $[\mathbf{D^n}\log det_\zeta\left(\mathbf{D^*D}\right)]\in\mathcal{D}'(M^n)$*) denotes the Schwartz kernel of the n-th differential* $D^n\log\det_\zeta(\Delta)$ *(resp* $D^n\log\det_\zeta(D^*D)$*).*

The choice of branch of the log is dictated by the Agmon angle but the results on the differentials of $\log\det_\zeta$ does not depend on the chosen branch of log.

There are several consequences of the above result. The first straightforward consequence is that the zeta determinants of Theorem 2 vanish exactly over non-invertible elements in $\mathcal{A}$ in the following sense:

$$\begin{aligned} \det_\zeta(\Delta+V)=0 &\Leftrightarrow \det_\zeta\left(\Delta+zV\right)=(z-1)^{\dim(\ker(\Delta+V))}\left(C+\mathcal{O}(z-1)\right), C\neq 0,\\ \det_\zeta(D^*(D+A))=0 &\Leftrightarrow \det_\zeta\left(D^*(D+zA)\right)=(z-1)^{\dim(\ker(D^*(D+A)))}\left(C+\mathcal{O}(z-1)\right), C\neq 0.\end{aligned}$$

Furthermore:

[8] Beware that $Q$ is not a local polynomial functional

**Corollary 3.1** (Zeta determinant for non smooth, non-self-adjoint perturbations)**.** *The zeta determinants of Theorem 2 extend as entire functions of* ***non smooth, non-self-adjoint*** *perturbations*

- *of* $\Delta$ *of regularity* $C^{d-3}(End(E))\cap L^{\infty}(End(E))$ *in the bosonic case,*
- *of* $D$ *of regularity* $C^{d-1}(Hom(E_+,E_-))$ *in the fermionic case.*

3.2. **An analytic reformulation of Quillen's conjectural picture.** In our setting, we attempt to reformulate Quillen's question as a problem of constructing an entire function with prescribed zeros in the infinite dimensional space $\mathcal{A}$ generalizing the Fredholm determinant. Our first Theorem 2 seems to indicate that the zeta determinant $\det_\zeta$ is a good candidate, but is it the only possible construction ? A naive approach suggested by Quillen in [61] would be to consider the Fredholm determinant $\det_F\left(Id+D^{-1}A\right)$ where for small $A$, we expect that

$$\log\det_F\left(Id+D^{-1}A\right)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}Tr\left((D^{-1}A)^k\right).$$

However as remarked by Quillen, the operator $D^{-1}A$ is a pseudodifferential operator of order $-1$, hence for $k>d$ the power $(D^{-1}A)^k$ is trace class hence the traces $Tr\left((D^{-1}A)^k\right)$ are well–defined whereas for $k\leqslant d$ these traces are ill–defined and often **divergent** as usual in QFT. We will later see how to deal with these divergent traces in Theorem 3.

We next formulate the general problem of finding renormalized determinants with functional properties closed to zeta determinants:

**Problem 3.2** (Renormalized determinants)**.** *Under the geometric setting from paragraph 2.4, set* $\mathcal{A}=P+C^\infty(Hom(E,F)),p=\deg(P)$ *where* $P=\Delta,p=2,E=F$ *in the bosonic case and* $P=D,p=1,E=E_+,F=E_-$ *in the fermionic case. An entire function* $\mathcal{R}\det:\mathcal{A}\mapsto\mathbb{C}$ *will be called renormalized determinant if*

(1) $\mathcal{R}\det$ *vanishes exactly on the subset of noninvertible elements in the following sense*

$$\mathcal{R}det(P+\mathcal{V})=0\Leftrightarrow\mathcal{R}det\left(P+z\mathcal{V}\right)=(z-1)^{\dim(\ker(P+\mathcal{V}))}\left(C+\mathcal{O}(z-1)\right),C\neq0,$$

*and* $\mathcal{R}\det$ *satisfies the bound:*

$$\boxed{\left|\mathcal{R}\det\left(P+\mathcal{V}\right)\right|\leqslant Ce^{K\|\mathcal{V}\|_{C^m}^{[\frac{d}{p}]+1}}}\tag{3.3}$$

*for the continuous norm* $\|.\|_{C^m}$ *on* $C^\infty(Hom(E,F))$ *where* $m=d-3$ *in the bosonic case,* $m=d-1$ *in the fermionic case and* $C,K>0$ *independent of* $\mathcal{V}$*.*

(2) *For* $n>[\frac{d}{p}]$*,*

$$\boxed{\frac{(-1)^{n-1}}{n-1!}\left(\frac{d}{dz}\right)^n\log\mathcal{R}\det(P+z\mathcal{V})|_{z=0}=Tr_{L^2}\left(\left(P^{-1}\mathcal{V}\right)^n\right).}\tag{3.4}$$

(3) *For* $\|\mathcal{V}\|_{C^m}$ *small enough, we further impose two conditions of microlocal nature on the second Gâteaux differential of* $\mathcal{R}\det$*. The first one reads:*

$$\boxed{D^2\log\mathcal{R}\det\left(P+\mathcal{V},V_1,V_2\right)=Tr_{L^2}\left(\left(P+\mathcal{V}\right)^{-1}V_1\left(P+\mathcal{V}\right)^{-1}V_2\right)}\tag{3.5}$$

*if supp*$(V_1)\cap$*supp*$(V_2)=\emptyset$ *where the* $L^2$ *trace is well–defined since* $(P+\mathcal{V})^{-1}V_1(P+\mathcal{V})^{-1}V_2$ *is smoothing.*
*Recall* $[\mathbf{D^2}\log\mathcal{R}\det(\boldsymbol{\Delta}+\mathcal{V})]\in\mathcal{D}'(M^2,Hom(E,F)\boxtimes Hom(E,F))$ *denotes the Schwartz kernel of the bilinear map* $D^2\log\mathcal{R}\det(P+\mathcal{V},.,.)$*, then the second condition reads:*

$$\boxed{WF\left([\mathbf{D^2}\log\mathcal{R}\det(\boldsymbol{\Delta}+\mathcal{V})]\right)\cap T^\bullet_{d_2}M^2\subset N^*\left(d_2\subset M^2\right).} \tag{3.6}$$

Note that in the fermionic case, our discussion is non trivial if the Fredholm index of $D$ vanishes. But in fact, we require in definition 2.7 that $D$ is invertible which means having Fredholm index 0 and $\ker(D)=\{0\}$ which is a stronger condition. Also the $L^2$ trace in the r.h.s. of equation 3.5 is well–defined since $(P+\mathcal{V})^{-1}V_1(P+\mathcal{V})^{-1}V_2$ is smoothing because the condition $\text{supp}(V_1)\cap\text{supp}(V_2)=\emptyset$ on the supports of $V_1,V_2$ implies the symbol of $(P+\mathcal{V})^{-1}V_1(P+\mathcal{V})^{-1}V_2$ vanishes (see [48, 1.1] for similar observations).

Let us motivate the axioms from definition 3.2. About condition 1), it is natural to require our determinants to vanish on noninvertible elements since they generalize the usual Fredholm determinant. Furthermore, we want to minimize the growth at infinity of the entire function $z\in\mathbb{C}\mapsto\mathcal{R}\det(P+z\mathcal{V})$, hence its *order* in the sense of subsection 2.1. We will see in corollary 11.2 that our condition on the order of $z\in\mathbb{C}\mapsto\mathcal{R}\det(P+z\mathcal{V})$ is optimal in the sense this is the smallest growth at infinity we can require. This is in some sense responsible for the polynomial ambiguity conjectured by Quillen which prevents us from having a unique solution to problem 3.2. In the same way, there is not necessarily a unique solution to the problem of finding an entire function with prescribed zeros in the Hadamard factorization Theorem 1.

About condition 2) that we impose on the derivatives of $\log\mathcal{R}\det$, this is reminiscent of the derivatives for the log of Gohberg–Krein's determinants $\mathcal{V}\mapsto\log\det_p\left(Id+P^{-1}\mathcal{V}\right)$. $\mathcal{V}\mapsto\det_p\left(Id+P^{-1}\mathcal{V}\right)$ also vanishes exactly on non-invertible elements. However, Gohberg–Krein's determinants $\mathcal{V}\mapsto\det_p\left(Id+P^{-1}\mathcal{V}\right)$ fail to satisfy the conditions on the second derivative of problem 3.2, hence by our main Theorem 3 they cannot be obtained from renormalization by subtraction of local counterterms since our Theorem 3 will show that these conditions are necessary to describe all renormalized determinants which can be obtained by a renormalization procedure where we subtract only local counterterms.

About condition 3), Equations (3.4) and (3.5) are very natural since they are reminiscent of the usual determinant in the finite dimensional case. In the seminal work of Kontsevich–Vishik [48, equation (1.4) p. 4], they attribute to Witten the observation that for the zeta determinant, the following identity

$$D^2\log\det_\zeta(A,A_1,A_2)=-Tr_{L^2}\left(A_1A^{-1}A_2A^{-1}\right)$$

holds true where $A_1,A_2$ are pseudodifferential deformations with **disjoint support**. This is not surprising provided we want our determinants to give rigorous meaning to QFT functional integrals. [9] Finally, we want to subtract only **smooth local counterterms in** $\mathcal{V}$, this smoothness will be imposed by the conditions on the wave front set of the Schwartz kernel

[9] In the present paper, we take this as axiom of our renormalized determinants and the identity (3.5) follows from a formal applications of Feynman rules.

of the Gâteaux differentials. The bound on $m$ is also optimal, locality forces renormalized determinants to depend on $m$-jets of the external potential $\mathcal{V}$.

3.2.1. *Solution of problem 3.2.* We now state the main Theorem of the present paper answering Problem 3.2, the assumptions are from paragraph 2.4 :

**Theorem 3** (Solution of the analytical problem). *A map $\mathcal{R}\det : \mathcal{A} \mapsto \mathbb{C}$ is a solution of problem 3.2 if and only if the following equivalent conditions are satisfied:*

(1) *there exists $Q \in \mathcal{O}_{loc,[\frac{d}{p}]}(J^m Hom(E,F))$ such that*

$$\begin{aligned} V \mapsto \mathcal{R}\det(\Delta+V) &= \exp\left(Q(V)\right) det_\zeta\left(\Delta+V\right), p=2, m=d-3 \text{ for bosons} \quad (3.7)\\ A \mapsto \mathcal{R}\det(D+A) &= \exp\left(Q(A)\right) det_\zeta\left(D^*(D+A)\right), p=1, m=d-1 \text{ for fermions.} \quad (3.8)\end{aligned}$$

(2) *$\mathcal{R}\det$ is renormalized by subtraction of local counterterms. There exists a generalized Laplacian $\Delta$ with heat operator $e^{-t\Delta}$ and a family $Q_\varepsilon \in \mathcal{O}_{loc,[\frac{d}{p}]}(J^m Hom(E,F)) \otimes_{\mathbb{C}} \mathbb{C}[\varepsilon^{-\frac{1}{2}}, \log(\varepsilon)]$ such that [10]:*

$$\mathcal{V} \mapsto \mathcal{R}\det\left(P+\mathcal{V}\right) = \lim_{\varepsilon\to 0^+} \exp\left(Q_\varepsilon(\mathcal{V})\right) det_F\left(Id + e^{-2\varepsilon\Delta}P^{-1}\mathcal{V}\right). \quad (3.9)$$

As immediate corollary of the above, we get that the group $\mathcal{O}_{loc,[\frac{d}{p}]}(J^m Hom(E,F))$ of local polynomial functionals acts freely and transitively on the set of renormalized determinants solutions to 3.2:

$$Q \in \mathcal{O}_{loc,[\frac{d}{p}]}(J^m Hom(E,F)) \mapsto \exp\left(Q(\mathcal{V})\right)\mathcal{R}\det\left(P+\mathcal{V}\right). \quad (3.10)$$

Theorem 3 also shows that zeta determinants are a particular case of some infinite dimensional family of renormalized determinants obtained by subtracting singular local counterterms.

**Corollary 3.3.** *In particular under the assumptions of Theorem 3 and using the same notations, $p=\deg(P)$ any function $\mathcal{R}\det\left(P+\mathcal{V}\right)$ can be represented as:*

$$\begin{aligned} \mathcal{R}\det\left(P+\mathcal{V}\right) &= \exp\left(Q(\mathcal{V})\right) det_{[\frac{d}{p}]+1}\left(Id+P^{-1}\mathcal{V}\right)\\ &= \exp\left(Q(\mathcal{V})\right)\prod_{n=1}^\infty E_{[\frac{d}{p}]}\left(\frac{1}{\lambda_n}\right)\end{aligned}$$

*where $Q$ is a polynomial functional of $\mathcal{V}$ of degree $[\frac{d}{p}]$, $\det_{[\frac{d}{p}]+1}$ is Gohberg–Krein's determinant, $E_k(z) = (1-z)e^{z+\frac{z^2}{2}+\dots+\frac{z^k}{k}}, k>0$ is a Weierstrass factor and the infinite product is over the sequence $\{\lambda | \dim\ker\left(P+\lambda\mathcal{V}\right) \neq 0\}$.*

[10] the choice of mollifier $e^{-2\varepsilon\Delta}$ is consistent with the GFF interpretation since the covariance of the heat regularized GFF $e^{-\varepsilon\Delta}\phi$ is $e^{-2\varepsilon\Delta}\Delta^{-1}$

3.3. **Renormalized determinants and holomorphic trivializations of Quillen's line bundle.** Let us quickly recall the notations from paragraph 1.0.3. For a pair $\mathcal{H}_0, \mathcal{H}_1$ of complex Hilbert spaces, there is a **canonical holomorphic line bundle** $\mathbf{Det} \longmapsto \text{Fred}_0(\mathcal{H}_0, \mathcal{H}_1)$ where $\text{Fred}_0(\mathcal{H}_0, \mathcal{H}_1)$ is the space of Fredholm operators of **index** 0 with fiber $\mathbf{Det}_B \simeq \Lambda^{top}\ker(B)^* \otimes \Lambda^{top}\text{coker}(B)$ over each $B \in \text{Fred}_0(\mathcal{H}_0, \mathcal{H}_1)$ and canonical section $\sigma$ [62, p. 32] [70, p. 137–138]. Recall in the fermionic situation, we considered the **complex affine space** $\mathcal{A} = D + C^\infty(Hom(E_+, E_-))$ of perturbations of some invertible, elliptic Dirac operator $D$. Then the map $\iota : D + A \in \mathcal{A} \mapsto Id + D^{-1}A \in \text{Fred}_0\left(L^2(E_+), L^2(E_+)\right)$ allows to pull–back the holomorphic line bundle $\mathbf{Det} \longmapsto \text{Fred}_0\left(L^2(E_+), L^2(E_+)\right)$ as a holomorphic line bundle $\mathcal{L} = \iota^*\mathbf{Det} \mapsto \mathcal{A}$ over the affine space $\mathcal{A}$ with canonical section $\underline{\det} = \iota^*\sigma$. We denote by $\mathcal{O}(\mathcal{L})$ the holomorphic sections from $\mathcal{L}$ and by $\mathcal{O}(\mathcal{A})$ holomorphic functions on $\mathcal{A}$.

**Theorem 4** (Holomorphic trivializations and flat connection)**.** *There is a bijection between the set of renormalized* $\mathcal{R}\det$ *from Theorem 3 and global holomorphic trivialization* $\tau : \mathcal{O}(\mathcal{L}) \mapsto \mathcal{O}(\mathcal{A})$ *of the line bundle* $\mathcal{L} \mapsto \mathcal{A}$ *such that*

$$T \in \mathcal{A} \mapsto \tau\left(\iota^*\underline{\det}(T)\right) = \mathcal{R}\det(T) \tag{3.11}$$

*is a solution of problem 3.2. The image of the canonical section* $\iota^*\underline{\det}(T)$ *under this trivialization being exactly the entire function* $\mathcal{R}\det$ *vanishing over non-invertible elements in* $\mathcal{A}$*.*

*For every pair* $(\tau_1, \tau_2)$*, there exists an element* $\Lambda$ *of the additive group* $(\mathcal{O}_{loc,d}, +)$ *s.t.*

$$\tau_1(D + A) = \exp\left(\int_M \Lambda(A(x))\right)\tau_2(D + A), \tag{3.12}$$

*where* $\Lambda$ *depends on the* $(d-1)$*-jet of* $A$*. For every choice of renormalized determinant* $\mathcal{R}\det$*, the section* $\sigma = \mathcal{R}\det^{-1}\iota^*\underline{\det}$ *defines a nowhere vanishing global holomorphic section with canonical holomorphic flat connection* $\nabla$ *s.t.* $\nabla\sigma = 0$*.*

The ambiguity group that relates all solutions of problem 3.2 is the renormalization group of Stüeckelberg–Petermann as described by Bogoliubov–Shirkov [5] and is interpreted here as a gauge group of the line bundle $\mathcal{L} \mapsto \mathcal{A}$. Our result is a variant of the so called main Theorem of renormalization by Popineau–Stora [60] and studied under several aspects by Brunetti–Fredenhagen [8] and Hollands–Wald [41, 42, 47]. In the aQFT community, there are various recent works exploring the renormalized Wick powers using Euclidean versions of the Epstein–Glaser renormalization [17, 18].

**Relation with other works.** The way we treat the problem of subtraction of local counterterms is strongly inspired by Costello's work [12] and the point of view of perturbative algebraic quantum field theory which is explained in Rejzner's book [64].

Perrot's notes [56] and Singer's paper [73] on quantum anomalies, which played an important role in our understanding of the topic, are in the real setting. The gauge potential $A$ which is used to perturb the half–Dirac operator preserves the Hermitian structure whereas we do not impose this requirement and view our perturbations as a complex space instead. Actually, our motivation to consider holomorphic determinants in some complexified setting bears strong inspiration from the work of Burghelea–Haller [9, 10] and Braverman–Kappeler [6] on finding some complex valued holomorphic version of the Ray–Singer analytic torsion.

In this short paragraph, we shall adopt the notation of [48]. Our renormalized determinants seem related to the multivalued function $f$ [48, prop 4.10] on the space of elliptic operators $Ell^m_{(-1)}(M;E,F)$ endowed with a natural complex structure [48, Remark 4.18] introduced in the seminal work of Kontsevich–Vishik. This multivalued function, defined on certain good classes of **non-self-adjoint operators**, naturally **extends the functional determinant** $\det_{\tilde{\pi}}$ defined on self-adjoint operators [48, Prop 4.12]. This is more general than the operators we consider in the present paper since we only work in the restricted class $\Delta+\Psi^0$ and $\text{Dirac}+\Psi^0$ where only the subprincipal part is allowed to vary whereas in [48] the full symbol is also allowed to change. We also note that [48, 4.1 p. 52] also consider determinants of Dirac operators exactly as in the present work but on odd dimensional manifolds. However, we obtain a factorization formula for $\det_\zeta$ in terms of Gohberg–Krein determinants and we relate zeta regularization with renormalization in quantum field theory which seem to be new results. Moreover, our factorization formula allows us to consider nonsmooth perturbations of generalized Laplacians or Dirac operators which also seems to be a new result. The way we define the determinant line bundle is very close to [48, section 6] and work of Segal and differs from the seminal work of Bismut–Freed [4] although all definitions should give the same object when restricted to self-adjoint families of Dirac operators. In particular, we do not focus on Quillen metrics and connections on the determinant line bundle which are important results of [4] but on the holomorphic structure instead and the relation with renormalization ambiguities as conjectured by Quillen in his notes [61, p. 284].

Finally, in a nice recent paper [26], Friedlander generalized the classical multiplicative formula $\det_\zeta(\Delta(Id+T)) = \det_\zeta(\Delta)\det_F(Id+T)$ when $T$ is smoothing, in [26, Theorem p. 4] connecting zeta determinants, Gohberg–Krein's determinants and Wodzicki residues. This bears a strong similarity with our Corollary 3.1 although our point of view stresses the relation with distributional extensions of products of Green functions[11] in configuration space. Another difference with his work is that we bound the wave front of the Schwartz kernels of the Gâteaux differentials of zeta determinants which is important from the QFT viewpoint and is related to the microlocal spectrum condition used in QFT.

## 4. Proof of Theorem 2.

We work under the setting of paragraph 2.4 and the zeta determinants are defined in definitions 2.8 and 2.9. We discuss in great detail the bosonic case for $\det_\zeta(\Delta+V)$ where $V\in C^\infty(M,End(E))$ and we indicate precisely the differences when we deal with the fermionic case for $\det_\zeta(\Delta+D^*A)$ where $\Delta=D^*D$ is a generalized Laplacian, the operator $D:C^\infty(E_+)\mapsto C^\infty(E_-)$ is a generalized Dirac operator and $A\in C^\infty(Hom(E_+,E_-))$. **Both cases consider zeta determinants of a non-self-adjoint perturbation of some**

[11] called *Feynman amplitudes* in physics litterature

**generalized Laplacian by some differential operator $V$ of order $0$ in the bosonic case and $V = D^*A$ of order $1$ in the fermionic case.**

### 4.1. Reformulation of the Theorem.

4.1.1. *Polygon Feynman amplitudes.* We will first state a reformulation of our Theorem 2 in terms of certain *Feynman amplitudes*. This emphasizes the relation of our result with quantum field theory as in the work of Perrot [56]. Before, we need to define these Feynman amplitudes as formal products of Schwartz kernels of the operators involved in our problem. This will play an important role in our Theorem:

**Definition 4.1** (Polygon Feynman amplitudes)**.** *Under the geometric setting from paragraph 2.4, we set $\mathbf{G} \in \mathcal{D}'(M \times M, E \boxtimes E^*)$ to be the Schwartz kernel of $\Delta^{-1}$ in the bosonic case and $\mathbf{G} \in \mathcal{D}'\left(M \times M, E_- \boxtimes E_+^*\right)$ is the Schwartz kernel of $D^{-1}$ in the fermionic case. For every $n \geqslant 2$, we formally set*

$$\boxed{t_n = \mathbf{G}(x_1, x_2) \ldots \mathbf{G}(x_{n-1}, x_n)\mathbf{G}(x_n, x_1)} \tag{4.1}$$

*which is well–defined in $C^\infty(M^n \setminus Diagonals)$.*

An important remark, we will later see in the proof of our main Theorem that the formal products $t_n$ are actually well–defined as distributions in $\mathcal{D}'(M^n \setminus d_n)$ where we deleted only the deepest diagonal $d_n$. This could also be easily proved by estimating the wave front set of the product $t_n = \mathbf{G}(x_1, x_2) \ldots \mathbf{G}(x_{n-1}, x_n)\mathbf{G}(x_n, x_1)$, using the fact that the wave front set of the pseudodifferential kernels are contained in the conormal bundle of the diagonal $d_2 \subset M \times M$.

Another remark related to quantum field theory. In QFT, formal products of Schwartz kernels, called *propagators*, are described by graphs [12] where every edge of a given graph stands for a Schwartz kernel $\mathbf{G}$ and vertices of the graph represent pointwise products of propagators. The correspondance Graph $\mapsto$ Distributional amplitude is described precisely in terms of the Feynman rules [16, 2.1]. Using the Feynman rules, one sees that the amplitude $t_n$ defined above represents some polygon graph with $n$ vertices and edges.

4.1.2. *Alternative statement of Theorem 2.* For any pair $(E, F)$ of Hermitian bundles over $M$, there is a natural fiberwise pairing $\langle t, \varphi\rangle$ between distributions $t$ in $\mathcal{D}'(M^n, Hom(F, E)^{\boxtimes n})$ with elements $\varphi$ in $C^\infty\left(M^n, Hom(E, F)^{\boxtimes n}\right)$ to get an element $\langle t, \varphi\rangle \in \mathcal{D}'(M^n)$. To obtain a number, we need to integrate this distribution against a density $dv \in |\Lambda^{top}|M^n$ as in $\int_{M^n} \langle t, \varphi\rangle \, dv$.

**Theorem 5.** *The zeta determinants from definition 2.9 are entire functions on $\mathcal{A}$ satisfying the factorization formula for $\|V\|_{C^{d-3}(End(E))}$ (resp $\|A\|_{C^{d-1}(Hom(E,F))}$) small enough:*

$$det_\zeta\left(\Delta + V\right) = e^{Q(V)} det_p\left(Id + \Delta^{-1}V\right), p = [\frac{d}{2}] + 1 \text{ in bosonic case} \tag{4.2}$$

$$det_\zeta(D^*(D + A)) = e^{Q(A)} det_p\left(Id + D^{-1}A\right), p = d + 1 \text{ in fermionic case} \tag{4.3}$$

[12] called Feynman graphs

*where* $\det_p$ *are Gohberg–Krein's determinants. There exists* $\ell \in C^\infty(End(E))$ *in bosonic case and* $\ell \in C^\infty(Hom(E_+, E_-))$ *in fermionic case s.t.*

$$Q(V) = \int_M \langle \ell, V \rangle \, dv + \sum_{2 \leqslant n \leqslant [\frac{d}{2}]} \frac{(-1)^{n+1}}{n} \int_{M^n} \left\langle \mathcal{R}t_n, V^{\boxtimes n} \right\rangle dv_n \tag{4.4}$$

$$Q(A) = \int_M \langle \ell, A \rangle \, dv + \sum_{2 \leqslant n \leqslant d} \frac{(-1)^{n+1}}{n} \int_{M^n} \left\langle \mathcal{R}t_n, A^{\boxtimes n} \right\rangle dv_n \tag{4.5}$$

- $dv \in |\Lambda^{top}|M$, $dv_n \in |\Lambda^{top}|M^n$ *are the canonical Riemannian densities,*
- $\mathcal{R}t_n$ *is a distribution of order* $m$ *on* $M^n$ *extending the distributional product* $t_n$ *from definition 4.1 which is well–defined on* $M^n \setminus d_n$, $m = d-3$ *in the bosonic case and* $m = d-1$ *in the fermionic case,*
- *the wave front set of* $\mathcal{R}t_n$ *satisfies the bound*

$$WF(\mathcal{R}t_n) \cap T^\bullet_{d_n} M^n \subset N^*(d_n \subset M^n).$$

Let us compare this statement with the original formulation of Theorem 2. The distribution $\mathcal{R}t_n$ in the above statement is nothing but the Schwartz kernel $[\mathbf{D^n} \log \det_\zeta(\boldsymbol{\Delta})]$ of the Gâteaux differentials up to some multiplicative constant.

4.1.3. *Plan of the proof.* The main idea of the proof of the Theorem in both bosonic and fermionic cases is to calculate Gâteaux differentials of $V \mapsto \log\det_\zeta(\Delta + V)$ with respect to the perturbation and compare with the Gâteaux differentials of some well chosen Gohberg–Krein determinant $V \mapsto \log\det_p\left(Id + \Delta^{-1}V\right)$. Physically, this seems to be a natural idea since we look for the response of the *free energy* $\log\det_\zeta$ (the logarithm of the partition function) under variation of the external field which is $V$ in the bosonic case and $A$ in the fermionic case. What we will prove is simply that starting from a certain order, all derivatives actually coincide and since we know that both sides are analytic, they must coincide up to some polynomial in the perturbation $V$. A second important idea is to recognize that the Schwartz kernels of the Gâteaux differentials are **distributional extensions** of products of Green kernels of the elliptic operator that we perturb, paired with external powers of the perturbation. This relies on explicit representation of the Schwartz kernels of Gâteaux differentials in terms of heat kernels.

The first step is to discuss the analyticity properties of $\det_\zeta$ in subsubsection 4.1.4. Our proofs rely on representations of complex powers and their differentials in terms of heat kernels in Lemma 4.12. This requires to consider small perturbations of $\Delta$ which maintain a spectral gap ensuring heat operators have exponential decay in subsection 4.2. But once the factorization formula is proved for small perturbations, it extends to $\mathcal{A}$ by analyticity of $\det_\zeta$. In subsection 4.6, we decompose the integral formula involving integrals of heat operators in two parts: a singular part involving short time of the heat operators that we control with the heat calculus of Melrose and a regular part involving the large time of the heat operators controlled by the exponential decay of the heat semigroup. This gives equation 4.19 representing differentials of $\det_\zeta$. Finally, the computation of differentials of $\log\det_\zeta$ for perturbations with disjoint supports in subsubsection 4.7.1 yields Proposition 4.16

which gives a characterization of the Schwartz kernels of $D^n \log \det_\zeta$ in terms of the Feynman amplitudes $t_n$ introduced in subsubsection 4.1.1. Furthermore, these results allow us to conclude the proof of the factorization formula of Theorem 2 in subsection 4.8. The bounds on the wave fronts are discussed in the next section 5.

4.1.4. *Analyticity.* The results of the present subsection are general enough to apply in both bosonic and fermionic cases.

Differential operators of order 1 on $M$ have a canonical structure of Fréchet space so it makes sense to talk about holomorphic curves in $\mathrm{Diff}^1(M, E)$. By [6, Thm 5.7 p. 215]:

**Proposition 4.2.** *For every holomorphic family $z \in \Omega \subset \mathbb{C} \mapsto V_z \in \mathit{Diff}^1(M, E)$ s.t. $\pi$ is an Agmon angle of $\Delta + V_z, \forall z \in \Omega$, in particular $\Delta + V_z$ is invertible and all $\theta \in [\pi - \varepsilon, \pi + \varepsilon]$ are principal angles of $\Delta + V_z$, the composition*

$$z \in \Omega \mapsto det_\zeta(\Delta + V_z) \in \mathbb{C}$$

*is holomorphic.*

In [48, 6], holomorphicity of the zeta determinant along one parameter holomorphic families of differential operators is established but where the full symbol is allowed to vary which is stronger than what is needed here. We now discuss how parametric holomorphicity implies holomorphicity for $\det_\zeta$ in the sense of definition 2.2. We next state a result proved in Kriegl–Michor [49, Thm 7.19 p. 88, Thm 7.24 p. 90] which gives a simple criteria which implies holomorphicity in the sense of definition 2.2.

**Proposition 4.3** (Holomorphicity by curve testing)**.** *Let $E$ be a Fréchet space over $\mathbb{C}$. Then given a map $F : E \mapsto \mathbb{C}$, the following statements are equivalent:*

- *For any holomorphic curve $\gamma : \mathbb{D} \subset \mathbb{C} \mapsto E$, the composite $F \circ \gamma : \mathbb{D} \mapsto \mathbb{C}$ is holomorphic in the usual sense,*
- *$F$ is holomorphic in the sense of definition 2.2.*

Then by Proposition 4.3, this implies that the maps $V \mapsto \det_\zeta(\Delta + V)$ in the bosonic case, and $A \mapsto \det_\zeta(\Delta + D^*A)$ in the fermionic case, are holomorphic for $V \in C^\infty(End(E))$ and $A \in C^\infty(Hom(E_+; E_-))$ close enough to 0.

4.2. **Perturbations with a spectral gap.** We need to consider small perturbations $\Delta + B$ of some positive definite generalized Laplacian $\Delta$ by differential operators $B$ of order 1 s.t. the corresponding heat semigroup $e^{-t(\Delta+B)}$ has exponential decay. Let us recall briefly some classical properties of such perturbations. The operator $\Delta + B : C^\infty(M) \mapsto H^s(M)$ admits a closed extension $H^{s+2}(M) \mapsto H^s(M)$ for every real $s \in \mathbb{R}$ by ellipticity of $\Delta + B \in \Psi^2(M)$. Since $(\Delta + B - z)^{-1}$ is compact and the resolvent set is non empty (see Lemma 4.4), by holomorphic Fredholm theory, the family $(\Delta + B - z)^{-1}$ is a meromorphic family of Fredholm operators with poles of finite multiplicity corresponding to the discrete spectrum of $\Delta + B : H^{s+2}(M) \mapsto H^s(M)$. In fact, it has been proved by Agranovich–Markus that such perturbations $\Delta + B$ have their spectrum contained in a parabolic region containing the real axis.

Our goal here is to bound the spectrum for small perturbations to show that the spectral gap of $\Delta$ is maintained. This will imply exponential decay of the semigroup $e^{-t(\Delta+B)}$. We assume that $\Delta$ is a positive, self-adjoint generalized Laplacian hence there is $\delta>0$ such that $\sigma(\Delta)\geqslant\delta$. We have the following Lemma proved in appendix:

**Lemma 4.4.** *There exists some neighborhood $\mathcal{U}$ of $0\in Diff^1(M,E)$ such that for every $B\in\mathcal{U}$, $\sigma(\Delta+B)\subset\{Re(z)\geqslant\frac{\delta}{2}, Im(z)\leqslant\frac{\delta}{2}\}$.*

So we control the spectrum of the perturbed operator in some $\frac{\delta}{2}$ neighborhood of the half–line $[\delta,+\infty)$.

Therefore, we can specialize the above Lemma to our particular situations as:

**Lemma 4.5.** *In the bosonic case, there exists some open neighborhood $\mathcal{U}\subset C^\infty(End(E))$ of $0$ such that for all small perturbations $V\in\mathcal{U}$, $\Delta+V$ is invertible and $\sigma(\Delta+V)\subset\{Re(z)\geqslant\frac{\delta}{2}\}$.*

*In the fermionic case, the discussion is similar. $D$ invertible implies that $\Delta=D^*D$ is positive self-adjoint, $\sigma(\Delta)\geqslant\delta>0$. It follows that there is some open subset $\mathcal{U}\subset C^\infty(Hom(E_+,E_-))$ s.t. $\forall A\in\mathcal{U}$, $\sigma(D^*(D+A))\subset\{Re(z)\geqslant\frac{\delta}{2}\}$.*

In the sequel, until subsection 4.8, we shall take small enough perturbations in the neighborhood $\mathcal{U}$ given by Lemma 4.5 so that the semigroups $e^{-t(\Delta+V)}$ and $e^{-t(D^*(D+A))}$ are analytic semigroups with exponential decay and the spectras $\sigma(\Delta+V)$ and $\sigma(D^*(D+A))$ are contained in the half–plane $Re(z)>0$.

4.2.1. *Taking the* $\log$ *of* $\det_\zeta$. In this situation, we can take the $\log\det_\zeta$. However our results on the functional derivatives of $\log\det_\zeta$ do not depend on the chosen branch of the logarithm. Since $\det_\zeta$ is well–defined for all perturbations and is holomorphic, all our identities on Gâteaux differentials that are proved for small perturbations will become automatically valid for any perturbations by analytic continuation.

4.3. **Resolvent bounds and exponential decay.** We also prove for small perturbations that we have some nice sectorial estimates on the resolvent which allow to apply the Hille–Yosida Theorem for analytic semigroups.

**Lemma 4.6.** *Let $B\in\mathcal{U}\subset Diff^1(M,E)$ where $\mathcal{U}$ is the open set from Lemma 4.4. Then there exists a convex angular sector $\mathcal{R}=\{\arg(z)\in[-\theta,\theta]\},\theta\in(0,\frac{\pi}{2})$ containing the half–line $[0,+\infty)$, $R>0$ s.t. the resolvent $(\Delta+B-z)^{-1}$ exists when $|z|\geqslant R, z\notin\mathcal{R}$ and satisfies a bound of the form:*

$$\|(\Delta+B-z)^{-1}\|_{\mathcal{B}(L^2,L^2)}\leqslant K\,dist(z,\mathcal{R}) \tag{4.6}$$

*for some $K>0$.*

These sectorial bounds are straightforward consequences of the more general [59, Theorem 9.2 p. 85 and Theorem 9.3 p. 86] since the operator $\Delta+B-z$ is elliptic with parameter $z$ of order 2 in the sense of Shubin [59, p. 79] on $M\times\Lambda$ where $\Lambda\subset\mathbb{C}$ is any closed cone containing $(-\infty,0]$ avoiding the half-line $[\frac{\delta}{2},+\infty)$.

A consequence of the above estimate also reads:

**Proposition 4.7.** *Under the assumptions of the previous Lemma, only a finite number of eigenvalues of $\Delta+B$ lies outside any conic neighborhood of the half–line $[0,+\infty)$.*

The resolvent bound on $(\Delta+B-z)^{-1}: L^2(M)\mapsto L^2(M)$ from Lemma 4.6 immediately implies by the Hille–Yosida theorem for analytic semigroups [39, Theorem 4.22 p. 36]:

**Proposition 4.8.** *Let $B\in\mathcal{U}\subset \mathit{Diff}^1(M,E)$ where $\mathcal{U}$ is the open set from Lemma 4.4. Then there exists a unique strongly continuous heat semigroup $e^{-t(\Delta+B)}: L^2(M)\mapsto L^2(M)$ generated by $\Delta+B$ which satisfies exponential decay estimates of the form*

$$\|e^{-t(\Delta+B)}\|_{\mathcal{B}(L^2,L^2)}\leqslant Ce^{-\frac{\delta}{2}t}.$$

4.4. **Relating the heat kernel and complex powers.** In the sequel, we will strongly rely on the formulation of complex powers in terms of the heat kernel. To make this correspondance precise, we shall need the following Proposition proved in appendix:

**Proposition 4.9.** *Let $B\in\mathcal{U}\subset \mathit{Diff}^1(M,E)$ where $\mathcal{U}$ is the open set from Lemma 4.4, for every $Q\in \mathit{Diff}^1(M,E)$, we have the following identity:*

$$\boxed{Tr_{L^2}\left(Q(\Delta+B)^{-s}\right)=\frac{1}{\Gamma(s)}\int_0^\infty Tr_{L^2}\left(Qe^{-t(\Delta+B)}\right)t^{s-1}dt.} \tag{4.7}$$

We shall use the above formula to represent differentials of $\log\det_\zeta$ in terms of heat operators in the next subsection.

4.5. **Gâteaux differentials.** Inspired by the nice exposition in Chaumard's thesis [11, p. 31-32], we calculate the derivatives in $z$ of $\log\det_\zeta(\Delta+zV)$ near $z=0$ and we find in the bosonic case that for $n>\frac{d}{2}$, the derivative of order $n$ of $z\mapsto\log\det_\zeta(\Delta+zV)$ at $z=0$ equals $(-1)^{n-1}(n-1)!Tr_{L^2}\left((\Delta^{-1}V)^n\right)$ where the $L^2$–trace is well–defined, in the fermionic case a similar result holds true for $n>d$.

We introduce a method which allows to simultaneously calculate the functional derivatives of $\log\det_\zeta$ and bound the wave front set of their Schwartz kernels. We start by using the following:

**Proposition 4.10.** *For any analytic family $(V_t)_{t\in\mathbb{R}^n}$ of perturbations, setting $A_t=\Delta+V_t$ we know that $Tr(\Delta+V_t)^{-s}$ is **holomorphic** near $s=0$ and depends smoothly on $t=(t_1,\dots,t_n)\in\mathbb{R}^n$, and satisfies the variation formula:*

$$\boxed{\frac{d}{dt_i}Tr\left(A_t^{-s}\right)=-sTr\left(\frac{dV}{dt_i}A_t^{-s-1}\right), i\in\{1,\dots,n\}} \tag{4.8}$$

*which is valid away from the poles of the analytic continuation in $s$ of $Tr_{L^2}\left(A_t^{-s}\right)$ hence the above equation holds true near $s=0$.*

*Proof.* In fact, the claim of our proposition is identical to [34, Theorem d) (1.12.2) p. 108] except Gilkey states his results **only for positive definite, self-adjoint** operators $\Delta+V$ whereas we need it for small non-self-adjoint perturbations $V\in C^\infty(End(E))$. We need to choose $V\in\mathcal{U}\subset C^\infty(End(E))$ where $\mathcal{U}$ is some sufficiently small neighborhood of 0 such that the semigroup $e^{-t(\Delta+V)}$ has exponential decay, $\mathcal{U}$ is given by Lemma 4.4. The proof in our non-self-adjoint case follows almost verbatim from Gilkey's proof since his proof relies on:

- the asymptotic expansion of the heat kernel for a smooth family of non-self-adjoint generalized Laplacians [34, Lemma 1.9.1 p. 75], the smoothness of the terms in the asymptotic expansion in the parameters and the variation formula for the resolvent and heat kernel which are proved in [34, Lemma 1.9.3 p. 77] in the non-self-adjoint case, we also refer to [3, section 2.7] for similar results,
- the result of [34, Lemma 1.12.1 p. 106] which still applies to $Tr_{L^2}(e^{-t(\Delta+V)})$,
- the identities

$$Tr_{L^2}\left(Q(\Delta+V)^{-s}\right)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}Tr_{L^2}\left(Qe^{-t(\Delta+V)}\right)dt, \forall Q\in \text{Diff}^1 \tag{4.9}$$

which we established in Proposition 4.9, and

$$Tr_{L^2}\left(Qe^{-t(\Delta+V)}\right)\sim\sum_n a_n(Q,\Delta+V)t^{\frac{n-\dim(M)-1}{2}} \tag{4.10}$$

which is established in [34, Lemma 1.9.1 p. 75].

□

The holomorphicity of $Tr(\Delta+V_t)^{-s}$ implies the Laurent series expansion $Tr(\Delta+V_t)^{-s}=\sum_{k=0}^\infty a_k(V_t)s^k$ near $s=0$. By definition $\log\det_\zeta(\Delta+V_t)=-\frac{d}{ds}|_{s=0}Tr(\Delta+V_t)^{-s}$ which implies that

$$\begin{aligned}\frac{d}{dt_i}\log\det_\zeta(\Delta+V_t) &= -\frac{d}{dt_i}\frac{d}{ds}|_{s=0}Tr(\Delta+V_t)^{-s}=-\frac{da_1(V_t)}{dt_i}\\ =\quad -\frac{d}{ds}|_{s=0}\sum_{k=0}^\infty\frac{da_k(V_t)}{dt_i}s^k &= -\frac{d}{ds}|_{s=0}\frac{d}{dt_i}Tr(\Delta+V_t)^{-s}=\frac{d}{ds}|_{s=0}sTr\left(\frac{dV}{dt_i}A_t^{-s-1}\right).\end{aligned}$$

Thus for higher derivatives, using equation (4.8), we immediately deduce that

$$\begin{aligned}\frac{d}{dt_1}\dots\frac{d}{dt_{n+1}}\log\det_\zeta(\Delta+V_t) &= \frac{d}{ds}|_{s=0}s\frac{d}{dt_1}\dots\frac{d}{dt_n}Tr\left(\frac{dV_t}{dt_{n+1}}A_t^{-s-1}\right)\\ &= FP|_{s=0}\frac{d}{dt_1}\dots\frac{d}{dt_n}Tr\left(\frac{dV_t}{dt_{n+1}}A_t^{-s-1}\right)\end{aligned}$$

where the finite part $FP$ of a meromorphic germ at $s=0$ is defined to be the constant term in the Laurent series expansion about $s=0$.

So specializing the above identity to the family $t\in\mathbb{R}^n\mapsto V+t_1V_1+\dots+t_{n+1}V_{n+1}$, we find a preliminary formula for the Gâteaux differentials of $\log\det_\zeta$ for bosons:

$$\boxed{D^{n+1}\log\det_\zeta(\Delta+V,V_1,\dots,V_{n+1})=FP|_{s=0}D^nTr\left((\Delta+V)^{-s-1}V_{n+1}\right)(V_1,\dots,V_n)} \tag{4.11}$$

and for fermions

$$\boxed{D^{n+1}\log\det_\zeta(\Delta+D^*A_0,A_1,\dots,A_{n+1})=FP|_{s=0}D^nTr\left((\Delta+D^*A_0)^{-s-1}D^*A_{n+1}\right)(A_1,\dots,A_n).}$$

The heavy notation $D^nTr\left((\Delta+V)^{-s-1}V_{n+1}\right)(V_1,\dots,V_n)$ means the $n$-th Gâteaux differential of the function $V\mapsto Tr\left((\Delta+V)^{-s-1}V_{n+1}\right)$ in the directions $(V_1,\dots,V_n)$.

At this level of generality the formulas in both bosonic and fermionic cases are very similar just replacing $V$ by $D^*A$ gives the fermionic formulas.

4.5.1. *Inverting traces and differentials.* There is a subtlety which motivates our next discussion. We would like to invert the Gâteaux differentials and the trace. In the next part, we shall prove estimates which allow to carefully justify the inversion of $Tr$ and Gâteaux differentials.

To calculate more explicitely the Gâteaux differentials on the r.h.s of equation 4.11, we shall study in more details the analytic map $V\mapsto(\Delta+V)^{-s-1}\in\mathcal{B}(L^2,L^2)$ in both bosonic and fermionic situations and try to represent this analytic map in terms of heat kernels.

4.5.2. *From the heat operator $e^{-t(\Delta+V)}$ to $(\Delta+V)^{-s-1}$ as analytic functions of $V$.* Assume $\Delta$ is a generalized Laplacian, not necessarily symmetric, s.t. $\sigma(\Delta)\subset\{Re(z)\geqslant\delta>0\}$. By Duhamel formula, the heat operator $e^{-t(\Delta+V)}$ can be expressed in terms of $e^{-t\Delta}$ as the Volterra series:

$$e^{-t(\Delta+V)}=\sum_{k=0}^{\infty}(-1)^k\int_{t\Delta_k}e^{-(t-t_k)\Delta}V\dots Ve^{-t_1\Delta} \tag{4.12}$$

where the series converges absolutely in $\mathcal{B}\left(L^2,L^2\right)$ since in the bosonic case, we have the bound

$$\|\int_{t\Delta_k}e^{-(t-t_k)\Delta}V\dots Ve^{-t_1\Delta}\|_{\mathcal{B}(L^2,L^2)}\leqslant e^{-t\delta}\frac{t^k\|V\|^k_{\mathcal{B}(L^2,L^2)}}{k!}.$$

Since $t\mapsto e^{-t\Delta}$ is a strongly continuous semigroup, it is easy to see that the series on the r.h.s of 4.12 is strongly continuous and defines a solution to the operator equation

$$\frac{dU}{dt}=-(\Delta+V)U(t):C^\infty(M)\mapsto L^2\text{ with }U(0)=Id \tag{4.13}$$

which defines uniquely the semigroup $e^{-t(\Delta+V)}$ hence this justifies that both sides are equal.

In the fermionic case, the convergence is slightly more subtle. We start from the bound

**Lemma 4.11.** *Assume that $\Delta=D^*(D+A_0)\in\mathcal{A}$ is a generalized Laplacian s.t. $\sigma(\Delta)\subset\{Re(z)\geqslant\delta>0\}$. For any differential operator $P$ of order* 1*,*

$$\|e^{-t\Delta}P\|_{\mathcal{B}(L^2,L^2)}\leqslant Ct^{-\frac{1}{2}}e^{-\frac{t}{2}\delta}. \tag{4.14}$$

*Proof.* Assume that $\Delta=D^*(D+A_0)\in\mathcal{A}$ is a generalized Laplacian s.t. $\sigma(\Delta)\subset\{Re(z)\geqslant\delta>0\}$ hence $\Delta$ is not necessarily self-adjoint. We shall use the following ingredient. In Proposition 4.9, we proved that fractional powers of $\Delta=D^*(D+A_0)$ defined as

$$\Delta^{-s}=\frac{1}{\Gamma(s)}\int_0^\infty e^{-t\Delta}t^{s-1}dt$$

coincide with complex powers of $\Delta$ defined by the contour integrals of the resolvent as in the work of Seeley. Note that by results of Seeley, the powers $\Delta^s, s\in\mathbb{R}$ are well–defined elliptic pseudodifferential operators of degree $2s$. Therefore

$$\|\Delta^{\frac{1}{2}}u\|_{L^2}=\|\Delta^{\frac{1}{2}}A^{-\frac{1}{2}}A^{\frac{1}{2}}u\|_{L^2}\leqslant\|\Delta^{\frac{1}{2}}A^{-\frac{1}{2}}\|_{\mathcal{B}(H^{-1},H^{-1})}\|A^{\frac{1}{2}}u\|_{L^2}=\|\Delta^{\frac{1}{2}}A^{-\frac{1}{2}}\|_{\mathcal{B}(H^{-1},H^{-1})}\|u\|_{H^{-1}},$$

where $\|\Delta^{\frac{1}{2}}A^{-\frac{1}{2}}\|_{\mathcal{B}(H^{-1},H^{-1})}<+\infty$ since $\Delta^{\frac{1}{2}}A^{-\frac{1}{2}}$ is a pseudodifferential operator of order 0 by composition. Recall that under our assumption of taking small perturbations of a positive definite self–adjoint $D^*D$, $e^{-t\Delta}$ generates an analytic semigroup.

$$\|Pe^{-t\Delta}\|_{\mathcal{B}(L^2,L^2)}\leqslant\underbrace{\|e^{-\frac{t}{2}\Delta}\|_{\mathcal{B}(L^2,L^2)}}_{\leqslant e^{-\frac{t}{2}\delta}}\|e^{-\frac{t}{2}\Delta}\Delta^{\frac{1}{2}}\|_{\mathcal{B}(L^2,L^2)}\underbrace{\|\Delta^{-\frac{1}{2}}P\|_{\mathcal{B}(L^2,L^2)}}$$

where the term underbraced is bounded by a constant since $\Delta^{-\frac{1}{2}}P\in\Psi^0$ by pseudodifferential calculus is bounded on every Sobolev space. We are now reduced to estimate the term $\|e^{-t\Delta}\Delta^{\frac{1}{2}}\|_{\mathcal{B}(L^2,L^2)}$. Then a straightforward application of [39, Proposition 4.36 p. 40], which is valid for generators of analytic semigroups whose spectrum is contained in the right half-plane, yields the estimate $\|e^{-t\Delta}\Delta^{\frac{1}{2}}\|_{\mathcal{B}(L^2,L^2)}\leqslant Ct^{-\frac{1}{2}}$ hence this implies the final result. □

Therefore setting $V=D^*A$, we find that the series on the r.h.s of identity (4.12) converges absolutely in $\mathcal{B}\left(L^2,L^2\right)$ by the bound

$$\begin{aligned}
&\|\int_{t\Delta_k}e^{-(t-t_k)\Delta}D^*A\dots D^*Ae^{-t_1\Delta}\|_{\mathcal{B}(L^2,L^2)}\\
\leqslant\ &\|A\|^k_{\mathcal{B}(L^2(E_+),L^2(E_-))}e^{-\frac{t}{2}\delta}C^k\int_0^t|t-t_k|^{-\frac{1}{2}}\dots\int_0^{t_2}|t_1|^{-\frac{1}{2}}dt_1\dots dt_k\\
=\ &\|A\|^k_{\mathcal{B}(L^2(E_+),L^2(E_-))}e^{-\frac{t}{2}\delta}\frac{(k+1)C^kt^k\Gamma(\frac{1}{2})^{k+1}}{\Gamma(\frac{k+3}{2})},
\end{aligned}$$

where $C$ is the constant of Lemma 4.11 and the r.h.s. is the general term of some convergent series by the asymptotic behaviour of the Euler $\Gamma$ function [13].

Furthermore using the Hadamard–Fock–Schwinger formula proved in Proposition 4.9, for $Re(s)>0$, we find that

$$\begin{aligned}
(\Delta+V)^{-s-1}V_{n+1}&=\sum_{k=0}^\infty\frac{(-1)^k}{\Gamma(s+1)}\int_0^\infty t^s\int_{t\Delta_k}e^{-(t-t_k)\Delta}V\dots Ve^{-t_1\Delta}V_{n+1}dt\text{ for bosons},\\
(\Delta+D^*A)^{-s-1}D^*A_{n+1}&=\sum_{k=0}^\infty\frac{(-1)^k}{\Gamma(s+1)}\int_0^\infty t^s\int_{t\Delta_k}e^{-(t-t_k)\Delta}D^*A\dots D^*Ae^{-t_1\Delta}D^*A_{n+1}dt\text{ for fermions},
\end{aligned}$$

where both series converge absolutely in $V\in\mathcal{B}\left(L^2(E),L^2(E)\right)$ (resp $A\in\mathcal{B}(L^2(E_+),L^2(E_-))$) by the above bounds since we have exponential decay in $t$.

[13] Rewrite $\int_0^t|t-t_k|^{-\frac{1}{2}}\dots\int_0^{t_2}|t_1|^{-\frac{1}{2}}dt_1\dots dt_k=\int_{\{u_1+\dots+u_{k+1}=t\}}\prod_{i=1}^{k+1}u_i^{-\frac{1}{2}}d\sigma$, $d\sigma$ measure on the simplex $\{u_1+\dots+u_{k+1}=t\}$. Note that $\int_{u_1+\dots+u_{k+1}=t}\prod_{i=1}^{k+1}u_i^{-\frac{1}{2}}d\sigma=\frac{d}{dt}\int_{\{u_1+\dots+u_{k+1}\leqslant t\}}\prod_{i=1}^{k+1}u_i^{-\frac{1}{2}}du_i=\frac{d}{dt}\frac{t^{k+1}\Gamma(\frac{1}{2})^{k+1}}{\Gamma(\frac{k+3}{2})}$ following a beautiful identity due to Dirichlet

From the above identities, we find that

$$D^{n+1}\log\det_\zeta(\Delta+V)(V_1,\dots,V_{n+1}) = FP|_{s=0}\frac{d}{dt_1}\dots\frac{d}{dt_n}Tr(\Delta+V_t)^{-s-1}V_{n+1})|_{t=0}$$

for the family $V_t = t_1V_1+\dots+t_nV_n$ and by definition of the Gâteaux differential.

$$\begin{aligned}
&\frac{d}{dt_1}\dots\frac{d}{dt_n}Tr((\Delta+V_t)^{-s-1}V_{n+1})|_{t=0}\\
=\ &\frac{d}{dt_1}\dots\frac{d}{dt_n}Tr(\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(s+1)}\int_0^\infty t^s\int_{t\Delta_k}e^{-(t-t_k)\Delta}V_t\dots V_te^{-t_1\Delta}V_{n+1}dt)|_{t=0}\\
=\ &\frac{(-1)^n n!}{\Gamma(s+1)}Tr(\int_0^\infty t^s\int_{t\Delta_n}e^{-(t-t_n)\Delta}V_1\dots V_ne^{-t_1\Delta}V_{n+1}dt).
\end{aligned}$$

[14] A similar identity holds true for Fermions.

**Lemma 4.12** (Gâteaux differentials of $\log\det_\zeta$)**.** *Following the notations from definitions (2.9) and (2.8). Let $M$ be a smooth, closed, compact Riemannian manifold of dimension $d$, and $\Delta$ (resp $\Delta = D^*D$) some generalized Laplacian acting on $E$ (resp $E_+$) s.t. $\sigma(\Delta)\subset\{Re(z)\geqslant\delta>0\}$ for bosons (resp fermions). The Gâteaux differentials of $\log\det_\zeta$ satisfy the following identities. For bosons, for every $(V_1,\dots,V_{k+1})\in L^\infty(M,End(E))^{k+1}$,*

$$\begin{aligned}
&\frac{1}{k!}D^{k+1}\log det_\zeta(\Delta+V,V_1,\dots,V_{k+1})|_{V=0}\\
=\ &FP|_{s=0}\frac{(-1)^k}{\Gamma(s+1)}Tr\left(\int_{[0,\infty)^{k+1}}(\sum_{e=1}^{k+1}u_e)^se^{-u_{k+1}\Delta}V_1\dots e^{-u_1\Delta}V_{k+1}\prod_{e=1}^{k+1}du_e\right).
\end{aligned}$$

*For fermions, for every $(A_1,\dots,A_{k+1})\in L^\infty(M,Hom(E_+,E_-))$*

$$\begin{aligned}
&\frac{1}{k!}D^{k+1}\log det_\zeta(\Delta+D^*A,A_1,\dots,A_{k+1})|_{A=0}\\
=\ &FP|_{s=0}\frac{(-1)^k}{\Gamma(s+1)}Tr\left(\int_{[0,\infty)^{k+1}}(\sum_{e=1}^{k+1}u_e)^se^{-u_{k+1}\Delta}D^*A_1\dots e^{-u_1\Delta}D^*A_{k+1}\prod_{e=1}^{k+1}du_e\right).
\end{aligned}$$

We want to determine more explicitely the above Gâteaux differentials and invert the $L^2$-trace and the integral. To justify this inversion, it suffices to prove that for $Re(s)>\frac{d+1}{2}$, for every integer $k$, the operator valued integral $\int_0^\infty t^s\int_{t\Delta_k}e^{-(t-t_k)\Delta}V\dots Ve^{-t_1\Delta}Vdt$ represents a trace class operator with continuous kernel. In this case, the $L^2$ trace of this operator coincides with the flat trace of this operator. Then it is enough to prove that the integral $\int_{[0,\infty)^{k+1}}(\sum_{e=1}^{k+1}u_e)^sTr_{L^2}\left(e^{-u_{k+1}\Delta}V_1\dots e^{-u_1\Delta}V_{k+1}\right)\prod_{e=1}^{k+1}du_e$ converges absolutely by carefully estimating the operator traces $Tr_{L^2}\left(e^{-u_{k+1}\Delta}V_1\dots e^{-u_1\Delta}V_{k+1}\right)$. We use the fact that outside some subset of measure 0, the operator $\left(e^{-u_{k+1}\Delta}V_1\dots e^{-u_1\Delta}V_{k+1}\right)$ is smoothing and depends continuously on $(u_1,\dots,u_{k+1})$ therefore both $L^2$ and flat trace coincide.

[14]The combinatorial factor $n!$ comes from the fact that for every symmetric polynomial $S_n: E^n\mapsto\mathbb{C}$ homogeneous of degree $n$ and $v_t = t_1v_1+\dots+t_nv_n; t=(t_1,\dots,t_n)\in\mathbb{R}^n$, $\frac{d}{dt_1}\dots\frac{d}{dt_n}S_n(v_t,\dots,v_t)|_{t=0} = n!S_n(v_1,\dots,v_n)$.

4.6. **Bounding operator traces.** Our next task is to make sense and study the analyticity properties of the integrals of the form

$$\int_{[0,\infty)^{k+1}} (\sum_{e=1}^{k+1} u_e)^s Tr\left(e^{-u_{k+1}\Delta}V_1 \dots e^{-u_1\Delta}V_{k+1}\right) \prod_{e=1}^{k+1} du_e = I(s, V_1, \dots, V_{k+1})$$

for $s$ near 0 and also determine the holomorphicity domain in $s$ as a function of $k \in \mathbb{N}$.

4.6.1. *A decomposition.* As in [16], the strategy relies on methods from quantum field theory: using the symmetries of the integrand by permutation of variables, we integrate on a simplex $\{u_{k+1} \geqslant \dots \geqslant u_1 \geqslant 0\}$ called Hepp's sector:

$$\begin{aligned} & \int_{[0,\infty)^{k+1}} (\sum_{e=1}^{k+1} u_e)^s e^{-u_{k+1}\Delta}V \dots e^{-u_1\Delta}V \prod_{e=1}^{k+1} du_e \\ = \quad & (k+1)! \int_{\{u_{k+1}\geqslant\dots\geqslant u_1\geqslant 0\}} (u_1 + \dots + u_{k+1})^s e^{-u_{k+1}\Delta}V \dots e^{-u_1\Delta}V du_1 \dots du_{k+1} \end{aligned}$$

We will show that the only divergence is in the variable $u_{k+1}$. In the next definition, we cut the integral in two parts, $u_{k+1} \geqslant 1$ and $u_{k+1} \leqslant 1$.

**Definition 4.13** (Decomposition)**.** *Under the assumptions of Lemma 4.12. We set*

$$\boxed{I(s, V_1, \dots, V_{k+1}) = \int_{[0,\infty)^{k+1}} (\sum_{e=1}^{k+1} u_e)^s Tr\left(e^{-u_{k+1}\Delta}V_1 \dots e^{-u_1\Delta}V_{k+1}\right) \prod_{e=1}^{k+1} du_e} \tag{4.15}$$

*that we shall decompose in two pieces*

$$I(s; V_1, \dots, V_{k+1}) = S(s; V_1, \dots, V_{k+1}) + R(s; V_1, \dots, V_{k+1}) \tag{4.16}$$

*where*

$$R(s; V_1, \dots, V_{k+1}) = \sum_{\sigma\in S_{k+1}} \int_{\{u_{k+1}\geqslant\dots\geqslant u_1\geqslant 0, u_{k+1}\geqslant 1\}} (\sum_{e=1}^{k+1} u_e)^s Tr\left(e^{-u_{k+1}\Delta}V_{\sigma(1)} \dots e^{-u_1\Delta}V_{\sigma(k+1)}\right) \prod_{e=1}^{k+1} du_e, \tag{4.17}$$

*and*

$$S(s; V_1, \dots, V_{k+1}) = \sum_{\sigma\in S_{k+1}} \int_{\{1\geqslant u_{k+1}\geqslant\dots\geqslant u_1\geqslant 0\}} (\sum_{e=1}^{k+1} u_e)^s Tr\left(e^{-u_{k+1}\Delta}V_{\sigma(1)} \dots e^{-u_1\Delta}V_{\sigma(k+1)}\right) \prod_{e=1}^{k+1} du_e. \tag{4.18}$$

We use the above decomposition for both bosons and fermions where $(V_i = D^*A_i, A_i \in C^\infty(Hom(E_+, E_-)))_{i=1}^{k+1}$ for fermions. The function $S$ (resp $R$) is the singular (resp regular) part of $I$. We will later deal with the singular part $S$ using the heat calculus of Melrose [53, Chapter 7] [37, 13].

4.6.2. *Controlling the regular part.* We shall first show that the regular part $R$ has analytic continuation as holomorphic function in $s$ on the whole complex plane.

**Lemma 4.14.** *Following the notations from definitions (2.9) and (2.8). Let $M$ be a smooth, closed, compact Riemannian manifold of dimension $d$, and $\Delta$ (resp $\Delta = D^*D$) some generalized Laplacian acting on $E$ (resp $E_+$) s.t. $\sigma(\Delta) \subset \{Re(z) \geqslant \delta > 0\}$ for bosons (resp fermions). For every $(V_1, \dots, V_{k+1}) \in C^\infty(End(E))^{k+1}$ in the bosonic case and for every $(A_1, \dots, A_{k+1}) \in C^\infty(Hom(E_+, E_-))$ where $V_1 = D^*A_1, \dots, V_{k+1} = D^*A_{k+1}$ in the fermionic case, the regular part $R(s; V_1, \dots, V_{k+1})$ has analytic continuation as a holomorphic function of $s \in \mathbb{C}$.*

*Proof.* For $p > d$ and $B \in \mathcal{B}(L^2, H^p)$, $B$ is trace class and satisfies the simple bound $|Tr_{L^2}(B)| \leqslant C\|B\|_{\mathcal{B}(L^2,H^p)}$ [25, Prop B 21 p. 502]. Hence in the bosonic case,

$$\begin{aligned} |Tr_{L^2}\left(e^{-u_{k+1}\Delta}V_1 \dots e^{-u_1\Delta}V_{k+1}\right)| &\leqslant \|e^{-\frac{1}{2}\Delta}\|_{\mathcal{B}(L^2,H^p)}\|e^{-(u_{k+1}-\frac{1}{2})\Delta}\|_{\mathcal{B}(L^2,L^2)}\prod_{i=1}^{k+1}\|V_i\|_{\mathcal{B}(L^2,L^2)} \\ &\leqslant e^{-(u_{k+1}-\frac{1}{2})\delta}\|e^{-\frac{1}{2}\Delta}\|_{\mathcal{B}(L^2,H^p)}\prod_{i=1}^{k+1}\|V_i\|_{\mathcal{B}(L^2,L^2)} \end{aligned}$$

the integrand has exponential decay which ensures the holomorphicity.

In the fermionic case where $(V_i = D^*A_i, A_i \in C^\infty(Hom(E_+, E_-)))_{i=1}^{k+1}$, the bound reads:

$$\begin{aligned} &|Tr_{L^2}\left(e^{-u_{k+1}\Delta}V_1 \dots e^{-u_1\Delta}V_{k+1}\right)| \\ \leqslant\ &\|e^{-\frac{1}{2}\Delta}\|_{\mathcal{B}(L^2,H^p)}\|e^{-(u_{k+1}-\frac{1}{2})\Delta}D^*\|_{\mathcal{B}(L^2,L^2)}\prod_{i=1}^{k}\|e^{-u_i\Delta}D^*\|_{\mathcal{B}(L^2,L^2)}\prod_{i=1}^{k+1}\|A_i\|_{\mathcal{B}(L^2,L^2)} \\ \leqslant\ &\sqrt{2}C^{k+1}\|e^{-\frac{1}{2}\Delta}\|_{\mathcal{B}(L^2,H^p)}e^{-(\frac{u_{k+1}}{2}-\frac{1}{4})\delta}\prod_{i=1}^{k}u_i^{-\frac{1}{2}}\prod_{i=1}^{k+1}\|A_i\|_{\mathcal{B}(L^2,L^2)} \end{aligned}$$

where $C$ is the constant from Lemma 4.11, $\|e^{-\frac{1}{2}\Delta}\|_{\mathcal{B}(L^2,H^p)} < +\infty$ since the heat kernel is smoothing and the r.h.s has exponential decay in $u_{k+1}$ which ensures holomorphicity. $\square$

4.6.3. *Bounding the singular part.* It remains to deal with the term $S(s; V_1, \dots, V_{k+1})$ involving the integral for $u_{k+1} \in [0, 1]$. Without loss of generality, we will discuss the case $V = V_1 = dots = V_{k+1}$ in the next two subsections, the general case follows by polarization.

*The case when $k = 0$.* In this simple case, for the bosonic case, we directly use the diagonal asymptotic expansion of the heat kernel [34, Lemma 1.8.2] $e^{-t\Delta}(x, x) \sim \sum_{k=0}^{\infty} \frac{a_k(x,x)t^{k-\frac{d}{2}}}{(4\pi)^{\frac{d}{2}}}$ which yields that $\int_0^{+\infty} u^s Tr\left(e^{-u\Delta}V\right)du$ is holomorphic when $Re(s) > \frac{d}{2} - 1$ since the integrand converges absolutely, it has analytic continuation as a meromorphic function in $s \in \mathbb{C}$

and also that

$$FP|_{s=0}\frac{1}{\Gamma(s+1)}\int_0^{+\infty}u^sTr\left(e^{-u\Delta}V\right)du = Tr\left(e^{-\Delta}\Delta^{-1}V\right)+\int_0^1 dt\int_M r_{N+1}(t,x,x)V(x)dv$$

$$+\sum_{k=0,k\neq\frac{d}{2}-1}^{N}\frac{\left(\int_M a_k(x,x)V(x)dv\right)}{(4\pi)^{\frac{d}{2}}\left(s+1+k-\frac{d}{2}\right)}-\frac{\Gamma'(1)}{(4\pi)^{\frac{d}{2}}}\left(\int_M a_{\frac{d}{2}-1}(x,x)V(x)dv\right).$$

This means there exists $\ell\in C^\infty\left(End(E)\right)$ and $dv\in|\Lambda^{top}|M$ a smooth density such that $FP|_{s=0}\frac{1}{\Gamma(s+1)}\int_0^{+\infty}u^sTr\left(e^{-u\Delta}V\right)du=\int_M\langle\ell,V\rangle dv$. A similar result holds true in the fermionic case where we find that $FP|_{s=0}\frac{1}{\Gamma(s+1)}\int_0^{+\infty}u^sTr\left(e^{-u\Delta}D^*A\right)du=\int_M\langle\ell,A\rangle dv$ where $\ell\in C^\infty(Hom(E_-,E_+))$ and $dv\in|\Lambda^{top}|M$.

*When* $k>0$. We use the formalism of the heat calculus of Melrose as exposed in the work of Grieser [37] (see also [77, p. 62] for related construction) whose notations are adopted. We start from the fact that in local coordinates, $e^{-t\Delta}(x,y)=t^{-\frac{d}{2}}\tilde{A}(t,\frac{x-y}{\sqrt{t}},y)$ where $\tilde{A}\in C^\infty\left([0,\infty)_{\frac{1}{2}}\times\mathbb{R}^d\times U,E\boxtimes E^*\right)$ since the heat kernel is an element in $\Psi_H^{-1}(M,E)$ [37, definition 2.1 p. 6]. Then, we note that the $k+1$-fold composition $K\star\dots\star K$ belongs to $\Psi_H^{-k-1}(M,E)$ by the composition Theorem in the heat calculus [37, Proposition 2.6 p. 8 ]. Hence this means for every $p\in M$, there are local coordinates $U\ni p$ s.t.:

$$K^{\star(k+1)}(t,x,y)=t^{-\frac{d+2}{2}+(k+1)}\tilde{A}(t,\frac{x-y}{\sqrt{t}},y)$$

where $\tilde{A}\in C^\infty\left([0,\infty)_{\frac{1}{2}}\times\mathbb{R}^d\times U,E\boxtimes E^*\right)$ by definition of the elements in the heat calculus $\Psi_H^\bullet(M,E)$. Therefore by definition of $\star$, we find:

$$S(s;V,\dots,V)=\int_0^1 t^s\int_{t\Delta_k}Tr\left(e^{-(t-t_k)\Delta}V\dots Ve^{-t_1\Delta}V\right)dt=\int_0^1 t^s\left(\int_M\left(K^{\star k+1}\right)(t,x,x)dv\right)dt$$

where $\int_0^1 t^s\left(K^{\star k+1}\right)(t,x,x)dt=\int_0^1 t^{s-\frac{d+2}{2}+(k+1)}\tilde{A}(t,0,x)dt$ in local coordinates on $M$ and the r.h.s is Riemann integrable in $t$ and holomorphic in the domain $Re(s)>\frac{d+2}{2}-k-2$ by Fubini. In particular, the term $S$ is holomorphic near $s=0$ as soon as $k+1>\frac{d}{2}$.

In the fermionic case, we start from the fact that in local coordinates, $e^{-t\Delta}(x,y)=t^{-\frac{d}{2}}\tilde{A}(t,\frac{x-y}{\sqrt{t}},y)$ where $\tilde{A}\in C^\infty\left([0,\infty)_{\frac{1}{2}}\times\mathbb{R}^d\times U,E_+\boxtimes E_+^*\right)$. From the observation that

$$D_{y^i}t^{-\frac{d}{2}}\tilde{A}(t,\frac{x-y}{\sqrt{t}},y)=t^{-\frac{d+1}{2}}(y^i-x^i)\left(D_{X^i}\tilde{A}\right)(t,\frac{x-y}{\sqrt{t}},y)+t^{-\frac{d}{2}}\left(D_{y^i}\tilde{A}\right)(t,\frac{x-y}{\sqrt{t}},y),$$

we deduce that $K=e^{-t\Delta}D^*A\in\Psi_H^{-\frac{1}{2}}(M,E_+)$. We loose $\frac{1}{2}$ of regularity compared to the bosonic case since we do not consider the heat kernel alone but composed with a differential operator of order 1. Then by composition in the heat calculus, we find that:

$$\begin{aligned}S(s;D^*A,\dots,D^*A) &= \int_0^1 t^s\int_{t\Delta_k}Tr\left(e^{-(t-t_k)\Delta}D^*A\dots e^{-t_1\Delta}D^*A\right)dt\\ &= \int_0^1 t^s\left(\int_M\left(K^{\star k+1}\right)(t,x,x)dv\right)dt\end{aligned}$$

where $\int_0^1 t^s \left(K^{\star k+1}\right)(t,x,x)dt = \int_0^1 t^{s-\frac{d+2}{2}+\frac{k+1}{2}} \tilde{A}(t,0,x)dt$ in local coordinates and the r.h.s is Riemann integrable in $t$, holomorphic in $s$ in the half–plane $Re(s) > \frac{d}{2} - \frac{k+1}{2}$ by Fubini and has analytic continuation as a meromorphic function in $s \in \mathbb{C}$. In particular, the term $S$ is holomorphic near $s = 0$ as soon as $k+1 > d$.

In both cases, when $k+1 > \frac{d}{2}$ in the bosonic case and $k+1 > d$ in the fermionic case, we find that

$$\lim_{s\to 0} I(s; V_1, \ldots, V_{k+1}) = \int_{[0,\infty)^{k+1}} Tr\left(e^{-u_{k+1}\Delta}V_1 \ldots e^{-u_1\Delta}V_{k+1}\right) \prod_{e=1}^{k+1} du_e$$

where the integral on the right hand side is convergent.

4.6.4. *Inverting integrals and traces in Lemma 4.12.* From the estimates on the operator trace $Tr\left(e^{-u_{k+1}\Delta}V_1 \ldots e^{-u_1\Delta}V_{k+1}\right)$ of Lemma 4.14 controlling the exponential decay for large times $u_{k+1} \in [1,+\infty)$ and of subsubsection 4.6.3 which bound the operator trace for small times $u_{k+1} \in [0,1]$, we conclude that the integrals

$$\int_{[0,\infty)^{k+1}} (\sum_{e=1}^{k+1} u_e)^s Tr\left(e^{-u_{k+1}\Delta}V_1 \ldots e^{-u_1\Delta}V_{k+1}\right) \prod_{e=1}^{k+1} du_e = I(s, V_1, \ldots, V_{k+1})$$

converge for $Re(s) > \frac{d}{2} + 1$. Therefore, this implies that we have the identity:

$$D^n\left(Tr\left((\Delta+V)^{-s-1} V_{n+1}\right), V_1, \ldots, V_n\right) = \frac{n!(-1)^n}{\Gamma(s+1)} I(s, V_1, \ldots, V_{n+1}). \tag{4.19}$$

Combined with the analytic continuation near $s = 0$ of both sides, this justifies the inversion of traces and integrals in the formula for the Gâteaux differentials of $\log\det_\zeta$ from Lemma 4.12.

4.7. **Schwartz kernels of the Gâteaux differentials of** $\log\det_\zeta$**.** Up to now, we worked with operators and their compositions. Now, we will work with integral kernels and products of operator kernels instead. The goal of the present subsection is to perform an in depth study of the Schwartz kernels $[\mathbf{D^n}\log\det_\zeta]$ of the $n$-th Gâteaux differentials $D^n\log\det_\zeta$ and to show they are distributional extensions of some products of Green functions (either the Schwartz kernel of $\Delta^{-1}$ for bosons or the Schwartz kernel of $D^{-1}$ for fermions), which is a priori well–defined only on $M^n \setminus d_n$, to the whole configuration space $M^n$.

4.7.1. *Gâteaux differentials with disjoint supports.* Let us start by discussing $[\mathbf{D^n}\log\det_\zeta]$ as a distribution on $M^n \setminus d_n$, so outside the deepest diagonal. In this subsubsection, we assume that $(V_1, \ldots, V_{k+1})$ are such that $\text{supp}(V_1) \cap \cdots \cap \text{supp}(V_{k+1}) = \emptyset$. So the mutual support is empty. We need the following Lemma whose proof is given in the appendix:

**Lemma 4.15** (Microlocal convergence of heat operator)**.** *Let $e^{-t\Delta}$ be the heat operator. Then we have the convergence $e^{-t\Delta} \underset{t\to 0^+}{\to} Id$ in $\Psi^{+0}_{1,0}(M)$.*

This implies that the family $(e^{-t\Delta}V_i)_{t\in[0,1]}$ defines a **bounded family of pseudodifferential operators** in $\Psi^{+0}_{1,0}(M,E)$ whose wave front set is uniformly controlled in $T^*_{\text{supp}(V_i)}M$.

This means that for every pair of cut–off functions $(\chi_1,\chi_2)\in C^\infty(M)^2$ s.t. $\text{supp}(V_i)\cap\text{supp}(\chi_1)\cap\text{supp}(\chi_2)=\emptyset$, the family $\left(\chi_2e^{-t\Delta}V_i\chi_1\right)_{t\in[0,+\infty)}$ is bounded in $\Psi^{-\infty}(M,E)$ [15]. Otherwise, $(\chi_2e^{-t\Delta}V_i\chi_1)_{t\in[0,+\infty)}$ is bounded in $\Psi^{+0}_{1,0}(M,E)$. This implies that the family

$$e^{-(t-t_k)\Delta}V_{k+1}\dots V_2e^{-t_1\Delta}V_1,\text{ for }\{0\leqslant t_1\leqslant\dots t_k\leqslant t\leqslant 1\}$$

is bounded in $\Psi^{-\infty}(M,E)$ by the condition on the support of $(V_i)_{i=1}^{k+1}$. Finally,

$$\int_{t\Delta_k}Tr\left(e^{-(t-t_k)\Delta}V_{k+1}\dots V_2e^{-t_1\Delta}V_1\right)=\mathcal{O}(1)$$

and $\lim_{s\to0}I(s,V_1,\dots,V_{k+1})=Tr_{L^2}\left(\Delta^{-1}V_1\dots\Delta^{-1}V_{k+1}\right)$ where the $L^2$-trace on the r.h.s is well–defined since $\Delta^{-1}V_1\dots\Delta^{-1}V_{k+1}\in\Psi^{-\infty}(M,E)$. Hence, for every $(k+1)$-uple of open subsets $(U_1,\dots,U_{k+1})$ s.t. $\overline{U_1}\cap\dots\cap\overline{U_{k+1}}=\emptyset$, the multilinear map

$$(V_1,\dots,V_{k+1})\in C^\infty_c(U_1,End(E))\times\dots\times C^\infty_c(U_{k+1},End(E))\mapsto\lim_{s\to0}\frac{I(s;V_1,\dots,V_{k+1})}{\Gamma(s+1)}=Tr\left(\Delta^{-1}V_1\dots\Delta^{-1}V_{k+1}\right)$$

is multilinear continuous and

$$(V_1,\dots,V_{k+1})\in C^\infty(End(E))^{k+1}\mapsto FP|_{s=0}\frac{I(s;V_1,\dots,V_{k+1})}{\Gamma(s+1)}$$

*coincides with the Gâteaux differential*

$$\frac{(-1)^k}{k!}D^{k+1}\log\det{}_\zeta(\Delta+V,V_1,\dots,V_{k+1})$$

of the analytic function $\log\det_\zeta$ on $C^\infty(End(E))$. Observe that $M^{k+1}\setminus d_{k+1}$ is covered by open subsets of the form $U_1\times\dots\times U_{k+1}$ s.t. $\overline{U_1}\cap\dots\cap\overline{U_{k+1}}=\emptyset$. By the multilinear Schwartz kernel (see definition 2.1), the above multilinear map is represented by a distribution $\mathcal{R}t_{k+1}\in\mathcal{D}'(M^{k+1},End(E)^{\boxtimes k+1})$ which **coincides** with the product $t_{k+1}=\mathbf{G}(x_1,x_2)\dots\mathbf{G}(x_{k+1},x_1)\in\mathcal{D}'(M^{k+1}\setminus d_{k+1},End(E)^{\boxtimes k+1})$ since

$$Tr\left(\Delta^{-1}V_1\dots\Delta^{-1}V_{k+1}\right)=\langle t_{k+1},V_1\boxtimes\dots\boxtimes V_{k+1}\rangle$$

for $\text{supp}(V_1)\cap\dots\cap\text{supp}(V_{k+1})=\emptyset$ and $\langle.,.\rangle$ is a distributional pairing on $M^{k+1}$. In the fermionic case, the discussion is almost identical.

From the above observation, we deduce the following claim which holds true in **both bosonic and fermionic settings** which summarizes all above results:

**Proposition 4.16.** *Following the notations from definitions (2.9) and (2.8). Let $M$ be a smooth, closed, compact Riemannian manifold of dimension $d$, and $\Delta$ (resp $\Delta=D^*D$) some generalized Laplacian acting on $E$ (resp $E_+$) s.t. $\sigma(\Delta)\subset\{Re(z)\geqslant\delta>0\}$ for bosons (resp fermions).*

*In the bosonic case, for every invertible $\Delta+V\in\mathcal{A}=\Delta+C^\infty(End(E))$, for every $(V_1,\dots,V_n)\in C^\infty(End(E))^n$, if $n>\frac{d}{2}$ or $supp(V_1)\cap\dots\cap supp(V_n)=\emptyset$ then:*

$$D^n\log det_\zeta(\Delta+V)(V_1,\dots,V_n)=(-1)^{n-1}(n-1)!Tr_{L^2}\left((\Delta+V)^{-1}V_1\dots(\Delta+V)^{-1}V_n\right).\tag{4.20}$$

[15] It means the corresponding family of Schwartz kernels are bounded in $C^\infty(M\times M,E\boxtimes E^*)$ for the usual Fréchet topology

*For general* $(V_1,\dots,V_n)\in C^\infty(End(E))^n$*:*

$$\frac{(-1)^{n-1}}{n-1!}D^n\log det_\zeta\left(\Delta+V\right)(V_1,\dots,V_n)=\langle\mathcal{R}t_n,V_1\boxtimes\dots\boxtimes V_n\rangle \tag{4.21}$$

*where* $\mathcal{R}t_n$ *is a* ***distributional extension*** *of* $t_n=\mathbf{G}(x_1,x_2)\dots\mathbf{G}(x_n,x_1)\in\mathcal{D}'(M^n\setminus d_n,End(E)^{\boxtimes n})$ *where* $\mathbf{G}\in\mathcal{D}'(M\times M,E\boxtimes E^*)$ *is the Schwartz kernel of* $(\Delta+V)^{-1}$*.*

*In the fermionic case, for every invertible* $D+A:C^\infty(E_+)\mapsto C^\infty(E_-)$*, for every* $(A_1,\dots,A_n)\in C^\infty(Hom(E_+,E_-))^n$*, if* $n>d$ *or* $supp\,(A_1)\cap\dots\cap supp\,(A_n)=\emptyset$ *then:*

$$D^n\log det_\zeta\left(\Delta+D^*A\right)(A_1,\dots,A_n)=(-1)^{n-1}(n-1)!Tr_{L^2}\left((D+A)^{-1}A_1\dots(D+A)^{-1}A_n\right). \tag{4.22}$$

*For general* $(A_1,\dots,A_n)\in C^\infty(Hom(E_+,E_-))^n$

$$\frac{(-1)^{n-1}}{n-1!}D^n\log det_\zeta\left(\Delta+D^*A\right)(A_1,\dots,A_n)=\langle\mathcal{R}t_n,A_1\boxtimes\dots\boxtimes A_n\rangle \tag{4.23}$$

*where* $\mathcal{R}t_n$ *is a* ***distributional extension*** *of* $t_n=\mathbf{G}(x_1,x_2)\dots\mathbf{G}(x_n,x_1)\in\mathcal{D}'(M^n\setminus d_n,Hom(E_-,E_+)^{\boxtimes n})$ *where* $\mathbf{G}\in\mathcal{D}'(M\times M,E_+\boxtimes E_-^*)$ *is the Schwartz kernel of* $(D+A)^{-1}$*.*

*Proof.* In the bosonic case, we proved the claim for all $\Delta+V$ s.t. $\sigma\left(\Delta+V\right)\subset\{Re(z)\geqslant\delta>0\}$ since we need the exponential decay of the heat semi–group $e^{-t(\Delta+V)}$ to make the regular part $R$ from definition 4.13 convergent. However, by **analyticity of both sides** of the identity $D^n\log\det_\zeta\left(\Delta+V\right)(V_1,\dots,V_n)=(-1)^{n-1}(n-1)!Tr_{L^2}\left(\left(\Delta+V\right)^{-1}V_1\dots\left(\Delta+V\right)^{-1}V_n\right)$ in $V\in C^\infty(M,End(E))$, the claim holds true everywhere on $C^\infty(M,End(E))$ by analytic continuation using the fact that the subset of invertible elements in $\mathcal{A}$ is **connected** [16]. The discussion is identical for the fermionic case. □

From the results of the above proposition, we can conclude the proof of Theorem 5 and prove all claims except the bound on the wave front set of the Schwartz kernels of the Gâteaux differentials for which we shall devote the whole section 5.

4.8. **Factorization formula relating** $\det_\zeta$ **and Gohberg–Krein's determinants** $\det_p$**.** We give the proof for bosons and write the factorization formula for fermions for simplicity since the discussion is almost similar in both cases. The proof relies crucially on the following well-known Lemma on the Gohberg–Krein determinants [71, Thm 9.2 p. 75]:

**Lemma 4.17** (Gohberg–Krein's determinants and functional traces)**.** *For all* $A\in\mathcal{I}_p$*, Gohberg–Krein's determinant* $\det_p(Id+zA)$ *is an* ***entire function*** *in* $z\in\mathbb{C}$ *and is related to traces* $Tr(A^n)$ *for* $n\geqslant p$ *by the following formulas:*

$$\boxed{det_p(Id+zA)=\exp\left(\sum_{n=p}^{\infty}\frac{(-1)^{n+1}z^n}{n}Tr(A^n)\right)=\prod_k\left[(1+z\lambda_k(A))\exp\left(\sum_{n=1}^{p-1}\frac{(-1)^n}{n}\lambda_k(A)^n\right)\right]}$$

*where the infinite product vanishes exactly when* $z\lambda_k(A)=-1$ *with multiplicity.*

[16] $\Delta+V$ invertible iff $Id+\Delta^{-1}V$ invertible which is true for $V$ in a small neighborhood of $V=0$. Then consider complex rays $z\in\mathbb{C}\mapsto Id+z\Delta^{-1}V$ which are non-invertible at isolated values of $z$ since $\Delta^{-1}V$ compact

The product $\prod_k \left[(1+z\lambda_k(A))\exp\left(\sum_{n=1}^{p-1}\frac{(-1)^n z^n}{n}\lambda_k(A)^n\right)\right]$ reads $\prod_k E_{p-1}(-z\lambda_k(A))$ where $E_{p-1}$ is the Weierstrass factor. From the above, we deduce:

**Proposition 4.18.** *Let $(M,g)$ be a closed compact Riemannian manifold of dimension $d$, $(E,F)$ a pair of isomorphic Hermitian bundles over $M$ and $P : C^\infty(E) \mapsto C^\infty(F)$ an invertible elliptic operator of degree $k$. For any $\mathcal{V} \in C^\infty(Hom(E,F))$, the series*

$$\sum_{n>\frac{d}{k}} \frac{(-1)^{n+1}}{n} Tr\left(\left(P^{-1}\mathcal{V}\right)^n\right)$$

*converges absolutely for $\|\mathcal{V}\|_{L^\infty(Hom(E,F))}$ small enough and*

$$\mathcal{V} \mapsto \exp\left(\sum_{n>\frac{d}{k}} \frac{(-1)^{n+1}}{n} Tr\left(\left(P^{-1}\mathcal{V}\right)^n\right)\right) = det_{[\frac{d}{k}]+1}\left(Id + P^{-1}\mathcal{V}\right)$$

*extends uniquely as an entire function on $C^\infty(Hom(E,F))$.*

*Proof.* Choose some auxiliary bundle isomorphism $E \mapsto F$ which induces an elliptic invertible operator $U \in \Psi^0(M,E,F) : L^2(E) \mapsto L^2(F)$ and $UP^{-1} \in \Psi^{-k}(M,E)$ belongs to the Schatten ideal $\mathcal{I}_{[\frac{d}{k}]+1}$ hence $\|UP^{-1}\|_{[\frac{d}{k}]+1} < +\infty$. The claim then follows from Lemma 4.17 applied to $A = P^{-1}\mathcal{V} \in \Psi^{-k}(M,E)$ which belongs to the Schatten ideal $\mathcal{I}_{[\frac{d}{k}]+1}$ and the series converges since the Schatten norm satisfies the estimate:

$$\|P^{-1}\mathcal{V}\|_{[\frac{d}{k}]+1} \leqslant \|U^{-1}\|_{\mathcal{B}(L^2(E),L^2(F))}\|\mathcal{V}\|_{\mathcal{B}(L^2(E),L^2(F))}\|UP^{-1}\|_{[\frac{d}{k}]+1}$$

which can be made $< 1$ if $\|\mathcal{V}\|_{L^\infty(Hom(E,F))} < \|UP^{-1}\|_{[\frac{d}{k}]+1}\|U^{-1}\|^{-1}_{\mathcal{B}(L^2(E),L^2(F))}$. □

Proposition 4.18 and Lemma 4.17 imply that for $z$ small enough, the series

$$\sum_{n\geqslant \frac{d}{2}+1} \frac{(-1)^{n+1}z^n}{n} Tr_{L^2}\left(\left(\Delta^{-1}V\right)^n\right)$$

converges and equals $\log\det_p\left(Id + z\Delta^{-1}V\right)$ for $p = [\frac{d}{2}]+1$. In particular, $\det_p\left(Id+\Delta^{-1}V\right)$ is analytic in $V \in C^\infty(End(E))$ and vanishes iff $\Delta+V$ is non-invertible and $z \mapsto \det_p\left(Id + z\Delta^{-1}V\right)$ is an entire function.

It follows from Proposition 4.16 that Gâteaux differentials of $\log\det_p(Id+\Delta^{-1}V)$ and $\log\det_\zeta(\Delta+V)$ at $V=0$ coincide when $k > \frac{d}{2}$ . Now we shall use the following general Lemma whose proof is given in subsection 11.4 in appendix:

**Lemma 4.19.** *Let $E$ be a complex Fréchet space and $F_1, F_2$ a pair of holomorphic functions on some open subset $\Omega \subset E$. Assume there is some integer $k > 0$, s.t. $F_1, F_2$ have the same Gâteaux differentials of order $n$ for every $n \geqslant k$. Then there exists a smooth polynomial function $P : E \mapsto \mathbb{C}$ of degree $k-1$ s.t. $F_1 = P + F_2$.*

The Lemma implies that we have the equality

$$\log\det_\zeta(\Delta+V) = P(V) + \log\det_p(Id+\Delta^{-1}V) \tag{4.24}$$

for $V$ close enough to $0$ where $P$ is a **continuous polynomial function** of $V$. Equation (4.24) together with the fact that $H\mapsto \det_p(Id+H)$ is an entire function on the Schatten ideal $\mathcal{I}_p$ vanishing exactly over noninvertible $Id+H$, proves that $V\mapsto \det_\zeta(\Delta+V)=e^{P(V)}\det_p(Id+\Delta^{-1}V)$ extends uniquely as an entire function on $\mathcal{A}$ vanishing exactly over noninvertible elements. Then by Proposition 4.16, the Schwartz kernels $\frac{(-1)^{n-1}}{n-1!}[\mathbf{D^n}\log\det_\zeta(\mathbf{\Delta})]$ of the Gâteaux differentials are **distributional extensions** of the distributions

$$t_n=\mathbf{G}(x_1,x_2)\dots\mathbf{G}(x_n,x_1)\in\mathcal{D}'(M^n\setminus d_n,End(E)^{\boxtimes n}).$$

It follows that $P(V)=\sum_{1\leqslant n\leqslant\frac{d}{2}}\frac{(-1)^{n+1}}{n}\left\langle\mathcal{R}t_n,V^{\boxtimes n}\right\rangle$ where $\mathcal{R}t_n\in\mathcal{D}'(M^n,End(E)^{\boxtimes n})$ is a distributional extension of the product $t_n\in\mathcal{D}'(M^n\setminus d_n,End(E)^{\boxtimes n})$.

It remains to estimate the wave front set of $\mathcal{R}t_n$ over the deepest diagonal $d_n$ which is the purpose of the next section where we will show that $WF\left(\mathcal{R}\left(t_n\right)\right)$ satisfies the bound $WF\left(\mathcal{R}\left(t_n\right)\right)\cap T^*_{d_n}M^n\subset N^*(d_n\subset M^n)$ by Proposition 5.4 in the next section. This will conclude the proof that $\det_\zeta$ admits the representation 4.4. The proof for fermions is similar and yields the factorization formula $\det_\zeta\left(\Delta+D^*A\right)=\exp\left(P(A)\right)\det_p\left(Id+\Delta^{-1}D^*A\right)$ for $p=d+1$ and $P$ a continuous polynomial of degree $d$ on $C^\infty(Hom(E_+,E_-))$.

Our goal for the next part is to study the microlocal properties of the Schwartz kernel $\mathcal{R}t_n=[\mathbf{D^n}\log\det_\zeta]$ near the deepest diagonal $d_n$. As explained in the introduction, the bounds on the wave fronts are needed to ensure the renormalized determinants are obtained by subtraction of smooth local counterterms.

## 5. Wave front set of Schwartz kernels of $D^n\log\det_\zeta$.

5.0.1. *Traces and integrals on configuration space.* First, we need to reformulate all composition of operators appearing in $D^n\log\det_\zeta$ as integrals of products of operator kernels on configuration space. In the bosonic case, for $(u_1,\dots,u_{k+1})\in(0,1]^{k+1}$, we reformulate the trace term $Tr\left(e^{-u_1\Delta}V_1\dots e^{-u_{k+1}\Delta}V_{k+1}\right)$ as an integral over configuration space

$$\int_{M^{k+1}}\left\langle e^{-u_1\Delta}(x_1,x_2)\dots e^{-u_{k+1}\Delta}(x_{k+1},x_1),\chi(x_1,\dots,x_{k+1})\right\rangle dv_{k+1}$$

where $dv_{k+1}\in|\Lambda^{top}|M^{k+1}$, the product $e^{-u_1\Delta}(x_1,x_2)\dots e^{-u_{k+1}\Delta}(x_{k+1},x_1)$ on the l.h.s is an element in $C^\infty(M^{k+1},End(E^*)^{\boxtimes k+1})$, $\chi=V\boxtimes\dots\boxtimes V\in C^\infty(M^{k+1},End(E)^{\boxtimes k+1})$ is the test section and the $\langle .,.\rangle$ denotes the natural fiberwise pairing between elements of $End(E)^{\boxtimes k+1}$ and $End(E^*)^{\boxtimes k+1}$. Starting from now on in the bosonic case, the test function part $\chi$ will be chosen arbitrarily in $C^\infty(M^{k+1},End(E)^{\boxtimes k+1})$.

In the fermionic case, we will consider the operator $e^{-t\Delta}D^*:C^\infty(E_-)\mapsto C^\infty(E_+)$ which has smoothing kernel when $t>0$ (since $\Psi^{-\infty}$ is an ideal) hence the trace term $Tr\left(e^{-u_1\Delta}D^*A_1\dots e^{-u_{k+1}\Delta}D^*A_{k+1}\right)$ is reformulated as the integral over configuration space:

$$\int_{M^{k+1}}\left\langle e^{-u_1\Delta}D^*(x_1,x_2)\dots e^{-u_{k+1}\Delta}D^*(x_{k+1},x_1),\chi(x_1,\dots,x_{k+1})\right\rangle dv_{k+1}\tag{5.1}$$

where $dv_{k+1}\in|\Lambda^{top}|M^{k+1}$, the product $e^{-u_1\Delta}D^*(x_1,x_2)\dots e^{-u_{k+1}\Delta}D^*(x_{k+1},x_1)$ on the l.h.s is an element in $C^\infty(M^{k+1},Hom(E_-,E_+)^{\boxtimes k+1})$, $\chi=A\boxtimes\dots\boxtimes A\in C^\infty(M^{k+1},Hom(E_+,E_-)^{\boxtimes k+1})$

and $\langle .,. \rangle$ denotes the natural fiberwise pairing between elements of $Hom(E_+, E_-)^{\boxtimes k+1}$ and $Hom(E_-, E_+)^{\boxtimes k+1}$.

In what follows, we will localize the study in some open subset of the form $U^{k+1}$ near an element of the form $(x, \dots, x) \in d_{k+1} \subset M^{k+1}$ where $U \subset M, x \in U$ is an open chart that we choose to identify with some bounded open subset $U$ of $\mathbb{R}^d$ making some abuse of notations. Recall that a consequence of the heat calculus is that in local coordinates $e^{-t\Delta}(x,y) = t^{-\frac{d}{2}}\tilde{A}(t, \frac{x-y}{\sqrt{t}}, y), \tilde{A} \in C^\infty([0,+\infty)_{\frac{1}{2}} \times \mathbb{R}^d \times U, E \boxtimes E^*)$ for bosons and $e^{-t\Delta}D^*(x,y) = t^{-\frac{d+1}{2}}\tilde{A}(t, \frac{x-y}{\sqrt{t}}, y), \tilde{A} \in C^\infty([0,+\infty)_{\frac{1}{2}} \times \mathbb{R}^d \times U, E_+ \boxtimes E_-^*)$ for fermions. From this observation on the asymptotics of the kernel $e^{-t\Delta}D^*$ the proofs in both bosonic and fermionic cases are uniform. The only changes occur in the numerology since there is a loss of $t^{-\frac{1}{2}}$ in powers of $t$ in the expansion of $e^{-t\Delta}D^*$.

**Definition 5.1.** *We define for* $(u,x) = ((u_e)_{e=1}^{k+1}, (x_i)_{i=1}^{k+1}) \in (0,1]^{k+1} \times U^{k+1}$*:*

$$J(u,x;\chi) = \left\langle \prod_{1 \leqslant e \leqslant k+1} \tilde{A}(u_e, \frac{x_{i(e)} - x_{j(e)}}{\sqrt{u_e}}, x_{j(e)}), \chi \right\rangle$$

*where* $i(e) = e, j(e) = e+1$ *when* $e \in \{1, \dots, k\}$ *and* $i(k+1) = k+1, j(k+1) = 1$*, the bracket* $\langle .,. \rangle$ *denotes the appropriate fiberwise pairing defined above,* $J(.,.,\chi) \in C^\infty\left((0,1]^{k+1} \times U^{k+1}\right)$ *and* $J$ *depends linearly on* $\chi$*.*

Then we can express $S$ from definition 4.13 in terms of $J$:

$$S(s;\chi) = \int_{\mathbf{\Delta}_{k+1}} (\sum_{e=1}^{k+1} u_e)^s \left(\int_{M^{k+1}} J(u,x;\chi)\right) d^{k+1}u, \chi = V_1 \boxtimes \cdots \boxtimes V_{k+1}.$$

We will next prove that $S$ is a distribution valued in meromorphic functions of the variable $s$ [16, def 3.5 p. 11] based on the methods of [16] using blow–ups. In fact, we will also recover the distributional order from our proof.

5.0.2. *Resolving products of heat kernels.* The problem in the definition of $S$ is that the product $\prod_{1 \leqslant e \leqslant k+1} \tilde{A}(u_e, \frac{x_{i(e)} - x_{j(e)}}{\sqrt{u_e}}, x_{j(e)})$ is not smooth on $\mathbf{\Delta}_{k+1} \times U^{k+1}$. We choose the test section $\chi$ supported in $U^{k+1} \subset M^{k+1}$. We integrate w.r.t the Riemannian volume $dv = \rho d^d x$ with smooth density $\rho$ w.r.t. the Lebesgue measure $d^d x$ hence without loss of generality, we choose to absorb $\rho$ in the test section $\chi$.

**Definition 5.2** (Blow–up)**.** *Consider the following change of variables:*

$$\begin{aligned} \beta : (x, h_1, \dots, h_k, t_1, \dots, t_{k+1}) &\longmapsto ((x_1 = x, x_i = x + \sum_{j=1}^{i-1} (t_j \dots t_{k+1}) h_j)_{i=2}^{k+1}, (u_l = \prod_{j=l}^{k+1} t_j^2)_{l=1}^{k+1}) \\ U \times \mathbb{R}^{kd} \times [0,1]^{k+1} &\longmapsto U^{k+1} \times \mathbf{\Delta}_{k+1} \end{aligned}$$

[16, def 5.3] *which resolves the singular product in* $J$*. We use the short notation* $(x,h,t) = (x, (h_1, \dots, h_k), (t_1, \dots, t_{k+1})) \in U \times \mathbb{R}^{kd} \times [0,1]^{k+1}$.

Replacing in the integral expression of $S$ yields,

$$S(s;\chi)=\int_{[0,1]^{k+1}\times U\times\mathbb{R}^{dk}}((t_1\dots t_k)^2+\dots+1)^s\beta^*J(t,x,h;\chi)t_1\dots t_{k+1}^{2k+2s+1}d^{k+1}td^dxd^{kd}h,$$

where the factor $t_{k+1}^{2k+1}$ comes from the Jacobian determinant $\beta^*\left(d^{k+1}u\right)=2^{k+1}t_1\dots t_{k+1}^{2k+1}d^{k+1}t$ in the change of variables. One of the key results from [16, Thm 5.2] is that for every $e\in\{1,\dots,k\}$, the pull–back $\beta^*\left(\frac{x_{i(e)}-x_{j(e)}}{\sqrt{u_e}}\right)$ by the blow–down map $\beta$ is a **smooth function** on the resolved space $U\times\mathbb{R}^{dk}$ hence $\beta^*J$ is also smooth on the resolved space. In the bosonic (resp fermionic) case, the change of variables in $\beta^*\left(\prod_{e=1}^{k+1}u_e^{-\frac{d}{2}}\right)$ (resp $\beta^*\left(\prod_{e=1}^{k+1}u_e^{-\frac{d+1}{2}}\right)$) brings a factor of the form $\prod_{1\leqslant l\leqslant k+1}(t_l\dots t_{k+1})^{-d}$ (resp $\prod_{1\leqslant l\leqslant k+1}(t_l\dots t_{k+1})^{-d-1}$) and the Jacobian determinant of the variable change yields a factor $(t_1\dots t_{k+1})^d\dots(t_kt_{k+1})^d$ from which we can extract the power of $t_{k+1}$ to be equal to $t_{k+1}^{-d}$ (resp $t_{k+1}^{-d-k-1}$). Replacing in the integral formula yields the following identity [16, Proposition 5.2],

$$\begin{aligned}S(s;\chi)=&\int_{[0,1]^{k+1}}((t_1\dots t_k)^2+\dots+1)^sP(t_1,\dots,t_k)\\&\times\left(\int_{\mathbb{R}^{d(k+1)}}A(t,x,h;\chi)d^dxd^{kd}h\right)t_{k+1}^{2s+2k+1-d}dt_1\dots dt_{k+1}\end{aligned}\tag{5.2}$$

where $P$ is a polynomial function whose explicit expression is irrelevant and $A(.;\chi)=\beta^*J(.;\chi)\in C^\infty([0,1]^{k+1}\times U\times\mathbb{R}^{dk})$. For fermions, we would get $t_{k+1}^{2s+k-d}$ in factor under the integral sign instead of $t_{k+1}^{2s+2k+1-d}$. As in [16, Lemma 5.4], $A$ depends **linearly** on $\beta^*\chi(t,x,h)=\chi(x,x+\sum_{j=1}^{i-1}(t_j\dots t_{k+1})h_j)_{i=2}^{k+1}$. We find

**Lemma 5.3.** *Under the previous notations, in both bosonic and fermionic case, the quantity* $\partial_{t_{k+1}}^pA(t,x,h)|_{t_{k+1}=0}$ *depends linearly on p-jets of the coefficients of* $\chi$ *along the diagonal* $d_{k+1}\subset M^{k+1}$.

In the following paragraph, we shall prove that both $\chi\mapsto S(s;\chi)$ and $\chi\mapsto I(s;\chi)$ are distributions valued in meromorphic germs at $s=0$.

5.0.3. *The case when* $1\leqslant k\leqslant\frac{d}{2}$ *(resp* $1\leqslant k\leqslant d$*) for bosons (resp fermions) case and integration by parts.* In the bosonic case, if $k\leqslant\frac{d}{2}$, the factor $t_{k+1}^{2(s+k)+1-d}$ appearing in factor of $A$ is potentially divergent since near $s=0$, $2(Re(s)+k)+1-d$ is no longer necessary $>-1$. Then as usual in Riesz regularization, we need to Taylor expand $A$ w.r.t the variable $t_{k+1}$ up to order $p$ in such a way that $(p+1)+2k+1-d>-1\implies p+1>d-2(k+1)$.

This yields

$$\begin{aligned}
&\int_{[0,1]^{k+1}}((t_1\dots t_k)^2+\dots+1)^s P(t_1,\dots,t_k)\\
\times\quad &\left(\int_{\mathbb{R}^{d(k+1)}} A(t,x,h;\chi)d^dxd^{kd}h\right)t_{k+1}^{2(s+k)+1-d}dt_1\dots dt_{k+1}\\
=\quad &\sum_{p=0}^{\sup(d-2(k+1),0)}\int_{[0,1]^k}((t_1\dots t_k)^2+\dots+1)^s P(t_1,\dots,t_k)\\
\times\quad &\frac{\int_{\mathbb{R}^{d(k+1)}}\partial^p_{t_{k+1}}A(t,x,h;\chi)|_{t_{k+1}=0}d^dxd^{kd}h}{s+k+p+1-\frac{d}{2}}dt_1\dots dt_k+\text{holomorphic at } s=0
\end{aligned}$$

where the holomorphic part depends linearly on the $(d-3)$-jet of $\chi$. This implies that in the general case $FP|_{s=0}\frac{1}{\Gamma(s+1)}I(s,\chi)$ depends linearly on the $(d-3)$-jets of $\chi$ and when $\text{supp}(\chi)$ does not meet the deepest diagonal $d_{k+1}\subset M^{k+1}$, we already know that

$$FP|_{s=0}\frac{1}{\Gamma(s+1)}I(s,\chi)=\int_{M^{k+1}}\left\langle\prod_{e=1}^{k+1}\mathbf{G}(x_{i(e)},x_{j(e)}),\chi\right\rangle\prod_{i=1}^{k+1}dv(x_i).$$

Altogether, this proves that the Schwartz kernel $\frac{(-1)^k}{k!}[\mathbf{D^{k+1}}\log\det_\zeta(\mathbf{\Delta})]$ is a **distributional extension** of the product $\mathbf{G}(x_1,x_2)\dots\mathbf{G}(x_{k+1},x_1)$. This distributional extension has **order at most** $(d-3)$. The fermionic case is similar only the numerology differs, we need to expand coefficients of $\chi$ at order $p$ in $t_{k+1}$ so that $p+1>d-2$ therefore $FP|_{s=0}\frac{1}{\Gamma(s+1)}I(s,\chi)$ depends linearly on $(d-1)$-jets of the coefficients of $\chi$.

5.0.4. *Bounds on the Fourier transform of the singular term $S$.* In this part, the bosonic and fermionic cases are similar and therefore we restrict to the former case for simplicity. To bound the wave front set of the meromorphic family of distributions $S(s;.)$, using the above notations, we should study $S(s;\chi)$ where the test function part $\chi$ is chosen of the form $\chi=\psi(x_1)e^{ix_1\xi_1}\dots\psi(x_{k+1})e^{ix_{k+1}\xi_{k+1}}$ where $\psi$ has small support in the coordinate chart $U$ of $M$ and for large momenta $(\xi_1,\dots,\xi_{k+1})$ in some closed conic set $V\subset\mathbb{R}^{d(k+1)*}$ that we will later determine. As usual for wave front bounds, we are just localizing with the function $\psi$ and taking Fourier transforms.

After the change of variables of definition 5.2, the exponential factor becomes

$$\exp\left(ix(\xi_1+\dots+\xi_{k+1})+i\sum_{e=2}^{k+1}\sum_{j=1}^{e-1}(t_j\dots t_{k+1})h_j\xi_e\right)$$
$$=\exp\left(ix(\xi_1+\dots+\xi_{k+1})+i\sum_{j=1}^{k}(t_j\dots t_{k+1})h_j(\sum_{j+1\leqslant e\leqslant k+1}\xi_e)\right).$$

Therefore, the term $A(t,x,h;\chi)$ has the explicit from

$$A(t,x,h;\chi)=A(t,x,h;\psi^{\boxtimes k+1})e^{\left(ix\sum_{j=1}^{k+1}\xi_j+i\sum_{e=2}^{k+1}\sum_{j=1}^{e-1}(t_j\dots t_{k+1})h_j\xi_e\right)}$$

so the interesting term in factor of $S$ that we should integrate by parts w.r.t. $t_{k+1}$ reads

$$\begin{aligned}
&|\partial_{t_{k+1}}^p \int_{\mathbb{R}^{d(k+1)}} e^{\left(ix\sum_{j=1}^{k+1}\xi_j + i\sum_{e=2}^{k+1}\sum_{j=1}^{e-1}(t_j\ldots t_{k+1})h_j\xi_e\right)} A(t,x,h;\psi^{\boxtimes k+1})d^dxd^{kd}h| \\
\leqslant\ & C(1+K\sum_{e=1}^{k+1}|\xi_e|)^p \sup_{1\leqslant j\leqslant p}\sup_{t\in[0,1]^{k+1}} |\widehat{\partial_t^j A}(t,\sum_{e=1}^{k+1}\xi_e,(t_j\ldots t_{k+1})\sum_{e=j+1}^{k+1}\xi_e;\psi^{\boxtimes k+1})| \\
\leqslant\ & C_N(1+|\sum_{e=1}^{k+1}\xi_e|)^{-N}
\end{aligned}$$

uniformly in $t\in[0,1]^{k+1}$ for all $N$ by **smoothness, in the $x$ variable, of $A$ and its derivatives $\partial_t^j A$**. Assume that $(\xi_1,\ldots,\xi_{k+1})$ belongs to some closed conic set $V\subset\mathbb{R}^{d(k+1)*}$ which does not meet the hyperplane $\{\sum_{i=1}^{k+1}\xi_i=0\}$. Then there exists a constant $C$ such that for every $(\xi_1,\ldots,\xi_{k+1})\in V$ satisfying $\sum_{i=1}^{k+1}|\xi_i|^2\geqslant R$, we have $|\sum_{i=1}^{k+1}\xi_i|\geqslant\varepsilon\left(1+\sum_{i=1}^{k+1}|\xi_i|\right)$. Therefore, we obtain the estimate:

$$\begin{aligned}
&|\partial_{t_{k+1}}^p \int_{\mathbb{R}^{d(k+1)}} e^{\left(ix\sum_{j=1}^{k+1}\xi_j + i\sum_{e=2}^{k+1}\sum_{j=1}^{e-1}(t_j\ldots t_{k+1})h_j\xi_e\right)} A(t,x,h;\psi^{\boxtimes k+1})d^dxd^{kd}h| \\
\leqslant\ & C_N\varepsilon^{-N}(1+\sum_{e=1}^{k+1}|\xi_e|)^{-N}
\end{aligned}$$

uniformly in $t\in[0,1]^{k+1}$. Finally, this means that for any element $(x,\ldots,x;\xi_1,\ldots,\xi_{k+1})$ in the wave front set $WF\left(FP|_{s=0}S(s,.)\right)$, we must have $\xi_1+\cdots+\xi_{k+1}=0$. This implies the wave front set bound

$$\left(WF\left(FP|_{s=0}S(s,.)\right)\cap T^\bullet_{d_{k+1}}M^{k+1}\right)\subset N^*\left(d_{k+1}\subset M^{k+1}\right)$$

over the diagonal $d_{k+1}$. In the next paragraph, we will use these bounds on the Fourier transform of $S$ to estimate the wave front set of the Schwartz kernel of the Gâteaux differentials over the diagonal.

5.0.5. *Wave front bounds.* The next step is to use the above methods to bound the wave front set of the Schwartz kernels of the Gâteaux differentials. An important application of the blow–up techniques is to estimate the wave front set of the extensions $\mathcal{R}t_n\in\mathcal{D}'(M^n)$ over the deep diagonal $d_n\subset M^n$ which is proved to be contained in the conormal bundle $N^*\left(d_n\subset M^n\right)$.

**Proposition 5.4.** *Following the notations from definitions (2.9) and (2.8). Let $M$ be a smooth, closed, compact Riemannian manifold of dimension $d$, $E\mapsto M$ some Hermitian bundle over $M$.*

*For every invertible generalized Laplacian $\Delta+V\in\mathcal{A}=\Delta+C^\infty(End(E))$ acting on $E$ s.t. $\sigma(\Delta)\subset\{Re(z)\geqslant\delta>0\}$, set $\mathbf{G}\in\mathcal{D}'(M\times M,E\boxtimes E^*)$ to be the Schwartz kernel of $(\Delta+V)^{-1}$. The Schwartz kernel of the Gâteaux differential of $\log\det_\zeta$ defined as $\mathcal{R}t_n=\frac{(-1)^{n-1}}{(n-1)!}[\mathbf{D^n}\log\det_\zeta(\mathbf{\Delta}+\mathbf{V})]$ is a distributional extension of the product*

$$t_n=\prod_{e\in E(G)}\mathbf{G}(x_{i(e)},x_{j(e)})\in\mathcal{D}'(M^n\setminus d_n,End(E^*)^{\boxtimes n})$$

*and satisfies the wave front set bound*

$$\boxed{\left(WF\left(\mathcal{R}t_n\right)\cap T^\bullet_{d_n}M^n\right)\subset N^*\left(d_n\subset M^n\right).}\tag{5.3}$$

*Proof.* We need to show that the distribution $\mathcal{R}t_n$ defined as

$$\langle \mathcal{R}t_n, V_1\boxtimes\dots\boxtimes V_n\rangle=\frac{(-1)^n}{(n-1)!}\frac{d}{ds}|_{s=0}D^nTr\left((\Delta+V)^{-s}\right)(V_1,\dots,V_n)$$

satisfies the wave front bound $WF(\mathcal{R}t_n)\cap T^\bullet_{d_n}M^n\subset N^*\left(d_n\subset M^n\right)$. In fact the proof reduces to the scalar case and we assume without loss of generality that we work with a scalar generalized Laplacian $\Delta$ acting on functions and the potential $V\in C^\infty(M,\mathbb{C})$. We start from the expression

$$\frac{1}{\Gamma(s+1)}\int_{[0,\infty)^n}\prod_{e=1}^n du_e(u_1+\dots+u_n)^s\left(\int_{M^n}e^{-u_1\Delta}(x_1,x_2)\dots e^{-u_n\Delta}(x_n,x_1)\chi(x_1,\dots,x_n)dv_n\right),$$

where $dv_n$ is the volume form on $M^n$. We work on a local chart $U^n$ where we choose the test section $\chi$ to be equal to $\chi=\psi(x_1)e^{ix_1\xi_1}\dots\psi(x_n)e^{ix_n\xi_n}$ where $\psi\in C^\infty_c(U)$ is supported on some chart $U$. There is a competition between:

(1) integration of heat kernels on $[1,+\infty)$ which yields smoothing operators in the sense the family $\left(e^{-u\Delta}(x,y)\right)_{u\in[1,+\infty)}$ is bounded in $C^\infty(M\times M)$ since $e^{-u\Delta}=e^{-\frac{1}{4}\Delta}\underbrace{e^{-(u-\frac{1}{2})\Delta}}_{\text{bounded}}e^{-\frac{1}{4}\Delta}$ where the term in the middle is uniformly bounded in $\mathcal{B}(L^2,L^2)$ by spectral assumption and both factors $e^{-\frac{1}{4}\Delta}$ on the left and right are smoothing operators in $(x,y)$ variable,

(2) integration on $[0,1]$ which yields singular distributions whose wave front set is conormal in the sense that the family $\left(e^{-u\Delta}(x,y)\right)_{u\in(0,1]}$ is a bounded family of distributions in $\mathcal{D}'_{N^*(d_2\subset M^2)}(M\times M)$ [17] which is the space of distributions whose wave front set is contained in the conormal bundle $N^*\left(d_2\subset M^2\right)$.

Introduce a first decomposition where we sum over permutations $S_n$ of $\{1,\dots,n\}$ in the second sum:

$$\int_{[0,\infty)^n}(u_1+\dots+u_n)^s\int_{M^n}\left\langle e^{-u_1\Delta}\dots e^{-u_n\Delta},\chi\right\rangle\prod_{e=1}^n du_e$$
$$=\sum_{k=0}^n\frac{1}{n!}\sum_{\sigma\in S_n}\int_{[0,1]^k\times[1,+\infty)^{n-k}}(u_1+\dots+u_n)^s\int_{M^n}\left\langle e^{-u_{\sigma(1)}\Delta}\dots e^{-u_{\sigma(n)}\Delta},\chi\right\rangle\prod_{e=1}^n du_e$$

Without loss of generality, we only treat the terms corresponding to the identity permutation of $S_n$. When $k<n$, we use the hypocontinuity of the product of distributions whose wave front set is fixed [15, Thm 6.1 p. 219]. From the fact that the family $e^{-u_i\Delta}(x_i,x_{i+1})$, viewed as distribution on $M^n$, is bounded in $\mathcal{D}'_{N^*\left(d_{\{i,i+1\}}\subset M^n\right)}(M^n)$ where $N^*\left(d_{\{i,i+1\}}\subset M^n\right)$ is the conormal of the diagonal $d_{\{i,i+1\}}=\{x_i=x_{i+1}\}\subset M^n$, we note that the distributional product $\left(e^{-u_1\Delta}(x_1,x_2)\dots e^{-u_k\Delta}(x_k,x_{k+1})\right)_{(u_1,\dots,u_k)\in[0,1]^k}$ is bounded in $\mathcal{D}'_\Gamma(M^n)$ for $\Gamma=\bigcup_I N^*\left(d_I\subset M^n\right)\subset T^\bullet M^n$, where the union runs over the sets $I=\{i,\dots,j\}$, where

[17] in the sense of the seminorms in [15, p. 204]

$\{i,\dots,j\}$ contains the arithmetic progression from $i$ to $j$, for $1\leqslant i<j\leqslant k$. Then it follows immediately that for $k<n$,

$$WF\left(\int_{[0,1]^k\times[1,+\infty)^{n-k}}(u_1+\dots+u_n)^s e^{-u_1\Delta}\dots e^{-u_n\Delta}\prod_{e=1}^n du_e\right)\cap T^*_{d_n}M^n\subset N^*\left(d_n\subset M^n\right).$$

For the term where $k=n$, the result follows simply from the bounds on the Fourier transform of the singular term $\left(WF(FP|_{s=0}S(s,.))\cap T^*_{d_n}M^n\right)\subset N^*\left(d_n\subset M^n\right)$ from paragraph 5.0.4. Gathering both cases yields the claim from the Proposition. $\square$

## 6. Proof of Theorem 3.

Equation 4.4 shows that zeta regularized determinants, defined by purely spectral conditions, admit a position space representation in terms of Feynman amplitudes and that zeta determinants are just a particular case of some infinite dimensional family of renormalized determinants obtained by subtraction of local counterterms.

As above, we give the proof for bosons since the fermion case is similar and presents no extra difficulties.

6.0.6. *Any element of the form $\mathcal{R}\det=e^{Polynomial}\det_\zeta$ solves Problem 3.2.* Assume $\mathcal{R}\det(\Delta+V)=e^{Q(V)}\det_\zeta(\Delta+V)$ for some $Q\in\mathcal{O}_{loc,[\frac{d}{2}]}(J^mEnd(E))$. The zeta determinants from definition 2.8 are solutions of problem 3.2 by Theorem 2 and Proposition 4.16 where we found the second Gâteaux differentials of $\det_\zeta$ to be equal to

$$D^2\log\det_\zeta(\Delta+V,V_1,V_2)=Tr_{L^2}((\Delta+V)^{-1}V_1(\Delta+V)^{-1}V_2)$$

when $(V_1,V_2)\in C^\infty(End(E))^2$ have disjoint supports and $\sigma(\Delta+V)\subset\{Re(z)\geqslant\delta>0\}$. Therefore, since $Q\in\mathcal{O}_{loc,[\frac{d}{2}]}(J^mEnd(E))$ is a local polynomial functional of degree $[\frac{d}{2}]$, the map $V\mapsto\mathcal{R}\det(\Delta+V)=\exp\left(Q(V)\right)\det_\zeta(\Delta+V)$ satisfies $D^2\log\mathcal{R}\det(\Delta+V,V_1,V_2)=D^2\log\det_\zeta(\Delta+V,V_1,V_2)$ where $D^2Q(V,V_1,V_2)=0$ since $(V_1,V_2)$ have disjoint supports and $Q$ is local [7, Prop V.5 p. 16]. This means

$$\begin{aligned}D^2\log\mathcal{R}\det(\Delta+V,V_1,V_2) &= D^2\log\det_\zeta(\Delta+V,V_1,V_2)\\ =Tr_{L^2}((\Delta+V)^{-1}V_1(\Delta+V)^{-1}V_2) &= \int_{M\times M}t_2(x_1,x_2)V(x_1)V(x_2)dv(x_1)dv(x_2).\end{aligned}$$

The wave front bound from Proposition 5.4 shows that the Schwartz kernel

$$[\mathbf{D^2}\log\mathcal{R}\det(\boldsymbol{\Delta}+\mathbf{V})]=[\mathbf{D^2}\log\det_\zeta(\boldsymbol{\Delta}+\mathbf{V})]+[\mathbf{D^2Q}(\mathbf{V})]$$

is a distribution $\mathcal{R}t_2\in\mathcal{D}'(M\times M)$ satisfying $WF\left(\mathcal{R}t_2\right)\cap T^\bullet_{d_2}M^2\subset N^*\left(d_2\subset M^2\right)$ where we used the fact that $WF\left([\mathbf{D^2Q}(\mathbf{V})]\right)\subset N^*\left(d_2\subset M^2\right)$ by [7, Lemma VI.9 p.19 ]. For the moment, we found $\mathcal{R}\det$ solves the equations (3.5) and (3.6) and equation (3.4) is easily satisfied by the factorization formula $\mathcal{R}\det(\Delta+V)=\det_\zeta\left(\Delta+V\right)e^{Q(V)}=e^{(P+Q)(V)}\det_{[\frac{d}{2}]+1}\left(Id+\Delta^{-1}V\right)$,

$\deg(P+Q)\leqslant [\frac{d}{2}]$ and the properties of $\det_p$. The last step is to use the factorization formula $\det_\zeta(\Delta+V)=e^{Q(V)}\det_{[\frac{d}{2}]+1}\left(Id+\Delta^{-1}V\right)$ from the previous section and the bound

$$|\det_{[\frac{d}{2}]+1}\left(Id+\Delta^{-1}V\right)|\leqslant e^{K_1\|\Delta^{-1}V\|^{[\frac{d}{2}]+1}_{[\frac{d}{2}]+1}}\leqslant e^{K_1\left(\|\Delta^{-1}\|_{[\frac{d}{2}]+1}\|V\|_{C^0}\right)^{[\frac{d}{2}]+1}}$$

which results from [71, b) Thm 9.2 p. 75] for the norm $\|.\|_{[\frac{d}{2}]+1}$ in the Schatten ideal $\mathcal{I}_{[\frac{d}{2}]+1}$, the fact that $\Delta^{-1}\in\Psi^{-2}(M,E)$ belongs to $\mathcal{I}_{[\frac{d}{2}]+1}$ since $\Delta^{-[\frac{d}{2}]-1}\in\mathcal{I}_1$ [25, Prop B.21] and Hölder's inequality $\|\Delta^{-1}V\|_{[\frac{d}{2}]+1}\leqslant\|\Delta^{-1}\|_{[\frac{d}{2}]+1}\|V\|_{C^0}$. From the above facts, we deduce the bound:

$$|\mathcal{R}\det(\Delta+V)|\leqslant|\det_\zeta(\Delta+V)||e^{P(V)}|\leqslant Ce^{K\|V\|^{[\frac{d}{2}]+1}_{C^m}}$$

for some $C,K>0$ independent of $V$ which proves the bound (3.3). Finally, $\mathcal{R}\det$ solves problem 3.2.

6.0.7. *Any renormalized determinant is of the form $e^{Polynomial}\det_\zeta$.* Let $\mathcal{R}\det$ be any other solution of problem 3.2, then for every $V$, the entire functions $z\mapsto\mathcal{R}\det(\Delta+zV)$ and $z\mapsto\det_\zeta(\Delta+zV)$ have the same divisor (which means same zeros with multiplicities). It follows that the ratio $z\mapsto\frac{\mathcal{R}\det(\Delta+zV)}{\det_\zeta(\Delta+zV)}$ is an entire function **without zeros** on $\mathbb{C}$ which satisfies the bound

$$\left|\frac{\mathcal{R}\det(\Delta+zV)}{\det_\zeta(\Delta+zV)}\right|\leqslant Ce^{K|z|^{[\frac{d}{2}]+1}\|V\|^{[\frac{d}{2}]+1}_{C^m}},m=d-3.$$

By the uniqueness part of Hadamard's Theorem 1, this implies that for every fixed $V$, $z\mapsto\frac{\mathcal{R}\det(\Delta+zV)}{\det_\zeta(\Delta+zV)}=e^{P(z;V)}$ where $P$ is a polynomial of degree $[\frac{d}{2}]+1$ in $z$. We already know the map $V\mapsto\log\mathcal{R}\det(\Delta+V)-\log\det_\zeta(\Delta+V)$ is analytic near $V=0$ hence locally bounded near $V=0$ and also the above shows that for every fixed $V$, $z\mapsto\log\mathcal{R}\det(\Delta+zV)-\log\det_\zeta(\Delta+zV)$ is a polynomial of degree $[\frac{d}{2}]+1$ in $z$. By proposition 11.5, this implies that the difference $\log\mathcal{R}\det(\Delta+V)-\log\det_\zeta(\Delta+V)=P(V)$ where $P$ is actually a **continuous polynomial function** in $V$ of degree $[\frac{d}{2}]+1$. But condition 3.4 imposes the derivatives $(\frac{d}{dz})^{[\frac{d}{2}]+1}\log\mathcal{R}\det(\Delta+zV)$ and $(\frac{d}{dz})^{[\frac{d}{2}]+1}\log\det_\zeta(\Delta+zV)$ to coincide at $z=0$ hence $P$ has degree $[\frac{d}{2}]$. It remains to show that $P$ is local. The fact that both $\log\mathcal{R}\det(\Delta+V)$ and $\log\det_\zeta(\Delta+V)$ are solutions of functional equation 3.5 implies that $D^2P(V,V_1,V_2)=0$ if $\text{supp}(V_1)\cap\text{supp}(V_2)=\emptyset$. Observe that

$$V\mapsto[\mathbf{D^2}\log\left(\frac{\mathcal{R}\det(\mathbf{\Delta}+\mathbf{V})}{\det_\zeta(\mathbf{\Delta}+\mathbf{V})}\right)]=[\mathbf{D^2P(V)}]\in\mathcal{D}'(M\times M)$$

is polynomial in $V$ valued in distributions on $M\times M$ with wave front set in $N^*\left(d_2\subset M^2\right)$. To extract the homogeneous part , we use the finite difference operator $\Delta_V$ defined in the proof of 11.5, the element $\frac{\Delta_V^{n-2}[\mathbf{D^2P(V)}]}{(n-2)!}=[\mathbf{P_n}(\mathbf{V},\dots,\mathbf{V},.,.)]\in\mathcal{D}'(M\times M)$ has also wave front set in $N^*\left(d_2\subset M^2\right)$ thus $V\mapsto[\mathbf{P_n}(\mathbf{V},\dots,\mathbf{V},.,.)]$ satisfies the assumptions of lemma 11.1 proved in appendix. Therefore $V\mapsto\left\langle\tilde{P}_n,V^{\boxtimes n}\right\rangle$ is a local functional which equals $\int_M\Lambda_n(V(x))dv(x)$ where $\Lambda_n(V(x))$ depends on the $m$–jets of $V$ at $x$ for $m=d-3$ and is homogeneous of degree

$n$ in $V$. It is important to stress that the function $\Lambda_n$ is not uniquely defined [18] but the **functional** $V \mapsto \int_M \Lambda_n(V(x))dv(x)$ is uniquely defined. Then locality of $P$ together with the representation formula for $\det_\zeta$ from Theorem 2 implies that any solution of problem 3.2 has the form given by equation 4.4. The infinite product representation is an easy consequence of the representation of Gohberg–Krein's's determinants $\det_p$ as infinite products.

To complete the proof of our Theorem, it remains to show that any $\mathcal{R}\det$ solution of Problem 3.2 is obtained by a renormalization with subtraction of local counterterms in the sense of the third property in 3 which is the goal of the next section.

## 7. Local renormalization and Theorem 7 on Gaussian Free Field representation.

We follow the notations from subsection 2.6 where we explained the notion of subtraction of local counterterms. The aim of this section is to show the third claim of Theorem 3. Namely that all $\mathcal{R}\det$ solutions from problem 3.2 are obtained from renormalization by subtraction of **local counterterms** which concludes the proof of Theorem 3: there exists a generalized Laplacian $\Delta$ with heat operator $e^{-t\Delta}$ and a family $Q_\varepsilon \in \mathcal{O}_{loc,[\frac{d}{p}]}\left(J^mHom(E_+,E_-)\right) \otimes_{\mathbb{C}} \mathbb{C}[\varepsilon^{-\frac{1}{2}},\log(\varepsilon)]$ such that:

$$\mathcal{V} \mapsto \mathcal{R}\det\left(P+\mathcal{V}\right) = \lim_{\varepsilon\rightarrow 0^+} \exp\left(Q_\varepsilon(\mathcal{V})\right)\det_F\left(Id+e^{-2\varepsilon\Delta}P^{-1}\mathcal{V}\right). \tag{7.1}$$

7.1. **Extracting singular parts.** In this subsection, we shall use the methods of [16] based on blow–ups to extract the singular parts of regularized traces $Tr_{L^2}\left(\left(e^{-2\varepsilon\Delta}\Delta^{-1}V\right)^n\right)$ to show:

**Lemma 7.1.** *In the bosonic case, for every $V \in C^\infty(M,End(E))$, we have an asymptotic expansion*

$$Tr_{L^2}\left(\left(e^{-2\varepsilon\Delta}\Delta^{-1}V\right)^{k+1}\right) = P_\varepsilon(V) + \mathcal{O}(1)$$

*where $P_\varepsilon(V) = \int_M \Lambda_\varepsilon(V)dv \in \mathcal{O}_{loc,[\frac{d}{2}]}\left(J^mEnd(E)\right)\otimes_{\mathbb{C}}\mathbb{C}[\varepsilon^{-\frac{1}{2}},\log(\varepsilon)]$ and $m=d-3$ and*

$$\begin{aligned}(V_1,\dots,V_{k+1}) \in C^\infty(End(E))^{k+1} &\mapsto FP|_{\varepsilon=0}Tr_{L^2}\left(e^{-2\varepsilon\Delta}\Delta^{-1}V_1\dots e^{-2\varepsilon\Delta}\Delta^{-1}V_{k+1}\right)\\ &= \lim_{\varepsilon\rightarrow 0^+} Tr_{L^2}\left(\left(e^{-2\varepsilon\Delta}\Delta^{-1}V\right)^{k+1}\right) - P_\varepsilon(V)\end{aligned}$$

*is well–defined and multilinear continuous.*

*For fermions, for every $A \in C^\infty(Hom(E_+,E_-))$ , we have an asymptotic expansion*

$$Tr_{L^2}\left(\left(e^{-2\varepsilon\Delta}\Delta^{-1}D^*A\right)^{k+1}\right) = P_\varepsilon(A) + \mathcal{O}(1)$$

*where $P_\varepsilon(A) = \int_M \Lambda_\varepsilon(A)dv \in \mathcal{O}_{loc,d}\left(J^mEnd(E)\right)\otimes_{\mathbb{C}}\mathbb{C}[\varepsilon^{-\frac{1}{2}},\log(\varepsilon)]$ and $m=d-1$ and*

$$\begin{aligned}(A_1,\dots,A_{k+1}) \in C^\infty(Hom(E_+,E_-))^{k+1} &\mapsto FP|_{\varepsilon=0}Tr_{L^2}\left(e^{-2\varepsilon\Delta}\Delta^{-1}D^*A_1\dots e^{-2\varepsilon\Delta}\Delta^{-1}D^*A_{k+1}\right)\\ &= \lim_{\varepsilon\rightarrow 0^+} Tr_{L^2}\left(\left(e^{-2\varepsilon\Delta}\Delta^{-1}D^*A\right)^{k+1}\right) - P_\varepsilon(A)\end{aligned}$$

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*is well–defined and multilinear continuous.*

Note that in the previous Lemma, the functionals $P_\varepsilon$ depend only on the $m$-jets of their argument.

*Proof.* We prove the claim only for bosons, the fermionic case is similar. In this lemma, we shall use the following notation, for two functions $a(\varepsilon), b(\varepsilon)$, we shall note $a \simeq b$ if $b-a = \mathcal{O}(1)$ when $\varepsilon \to 0^+$. This means that $a, b$ have the same singular parts as $\varepsilon$ approaches 0. We start from the identity:

$$Tr_{L^2}\left(e^{-2\varepsilon\Delta}\Delta^{-1}V_1 \dots e^{-2\varepsilon\Delta}\Delta^{-1}V_{k+1}\right) \simeq \int_{[\varepsilon,1]^{k+1}} Tr_{L^2}\left(e^{-u_1\Delta}V_1 \dots e^{-u_{k+1}\Delta}V_{k+1}\right) du_1 \dots du_{k+1}$$

as a direct consequence of $e^{-2\varepsilon\Delta}\Delta^{-1} = \int_{2\varepsilon}^\infty e^{-t\Delta}dt$ and since the operator valued integral $\int_1^\infty e^{-t\Delta}dt \in \Psi^{-\infty}$ is smoothing. Now without loss of generality and using the symmetry of the integral, we may assume that we work in the Hepp sector $\{\varepsilon \leqslant u_1 < \dots < u_{k+1} \leqslant 1\}$ which is a semialgebraic subset of the unit simplex $\mathbf{\Delta}_{k+1} = \{0 \leqslant u_1 \leqslant \dots \leqslant u_{k+1} \leqslant 1\}$. So we need to study the asymptotics when $\varepsilon \to 0^+$ of

$$(k+1)! \int_{\{\varepsilon\leqslant u_1<\dots<u_{k+1}\leqslant 1\}} Tr_{L^2}\left(e^{-u_1\Delta}V_1 \dots e^{-u_{k+1}\Delta}V_{k+1}\right) du_1 \dots du_{k+1}.$$

Setting $\chi = V_1\boxtimes\cdots\boxtimes V_{k+1} \in C^\infty(M^{k+1}, End(E)^{k+1})$ and using the notations and conventions from paragraphs 5.0.1 and 5.0.2, the blow–up from definition 5.2 yields a blow–down map

$$\begin{aligned} \beta : (x, h_1, \dots, h_k, t_1, \dots, t_{k+1}) &\longmapsto ((x_1 = x, x_i = x + \sum_{j=1}^{i-1}(t_j \dots t_{k+1})h_j)_{i=2}^{k+1}, (u_l = \prod_{j=l}^{k+1} t_j^2)_{l=1}^{k+1}) \\ U \times \mathbb{R}^{kd} \times \Omega_\varepsilon &\longmapsto U^{k+1} \times \{\varepsilon \leqslant u_1 < \dots < u_{k+1} \leqslant 1\} \end{aligned}$$

where $\Omega_\varepsilon$ is the **semialgebraic set** defined by $\Omega_\varepsilon = \{\varepsilon \leqslant (t_1 \dots t_{k+1})^2\} \cap [0,1]^{k+1}$. Now, following the calculations of paragraph 5.0.3 we set:

$$\omega(\chi) = \left(\int_{\mathbb{R}^{d(k+1)}} t_{k+1}^{2k+1-d} P(t_1, \dots, t_k)\beta^* J(t, x, h; \chi) d^d x d^{kd} h\right) dt_1 \wedge \dots \wedge dt_{k+1}$$

where $t_{k+1}^{d-2k-1}\omega$ is a smooth differential form of top degree on the cube $[0,1]^{k+1}$. To extract the singular part of $\int_{\Omega_\varepsilon}\omega$, we need to Taylor expand $\int_{\mathbb{R}^{d(k+1)}} A(t,x,h;\chi) d^d x d^{kd} h$ in the variable $t_{k+1}$:

$$\int_{\Omega_\varepsilon}\omega \simeq \sum_{j=0} \int_{\Omega_\varepsilon} \frac{t_{k+1}^{2k+1-d+j}}{j!} P(t_1, \dots, t_k) \underbrace{\left(\partial_{t_{k+1}}^j \int_{\mathbb{R}^{d(k+1)}} A(t,x,h;\chi) d^d x d^{kd} h\right)|_{t_{k+1}=0}} dt_1 \wedge \dots \wedge dt_{k+1}$$

where the term underbraced is a conormal distribution of $\chi \in C_c^\infty(U^{k+1}, End(E)^{\boxtimes k+1})$ supported by $d_{k+1}$ by the results of paragraph 5.0.4. So setting $\chi = V^{\boxtimes k+1}$, we can view the term underbraced as a functional of $V$: the map

$$V \in C_c^\infty(U, End(E)) \mapsto \left(\partial_{t_{k+1}}^j \int_{\mathbb{R}^{d(k+1)}} A(t,x,h;V^{\boxtimes k+1}) d^d x d^{kd} h\right)|_{t_{k+1}=0}$$

is an element of $\mathcal{O}_{loc,k+1}\left(J^j E\right)$.

Thus to extract precise asymptotics, we set

$$\omega_j = \frac{t_{k+1}^{2k+1-d+j}}{j!}\left(\partial_{t_{k+1}}^j P\int_{\mathbb{R}^{d(k+1)}} A(t,x,h;\chi)d^dxd^{kd}h\right)|_{t_{k+1}=0}dt_1\wedge\cdots\wedge dt_{k+1}.$$

Then we may slice the **semialgebraic set** $\Omega_\varepsilon$ by the fibers $(t_1\ldots t_{k+1})^2=$ constant of the map $F(t_1,\ldots,t_{k+1})=(t_1\ldots t_{k+1})^2$. Practically, this means we will pushforward the differential form $\omega_j$ along the fibers of the $b$-map $F:(t_1,\ldots,t_{k+1})\in[0,1]^{k+1}\mapsto F(t_1,\ldots,t_{k+1})\in\mathbb{R}$ [38, def 2.11 p. 16] [51, p. 51–52] where the cube $[0,1]^{k+1}$ is viewed as a $b$-manifold in the sense of Melrose [38, def 2.2 p. 8] [51, p. 51]. Then we will conclude by using the pushforward Theorem of Melrose [51, Thm 4 p. 58] in the form discussed in the nice survey of Grieser [38, Thm 3.6 p. 25]. We define the form $\frac{\omega_j}{dF}$ which is called Gelfand–Leray form [80, Lemma 5.11 p. 123] and the function $J_j(t)=\int_{F=t}\frac{\omega_j}{dF}$. By Fubini's Theorem, we find that $\int_{\Omega_\varepsilon}\omega_j=\int_\varepsilon^1 J_j(t)dt$. Finally, the pushforward Lemma 7.2, which is stated in the next subsubsection below, implies that for each $j\in\mathbb{N}$, the map $\varepsilon\mapsto\int_{\Omega_\varepsilon}\omega_j$ admits an asymptotic expansion as $\varepsilon\to0^+$ of the required form which concludes the proof. □

7.1.1. *Pushforward Lemma.* Here we state the Lemma on asymptotic integrals used in the previous proposition.

**Lemma 7.2** (Pushforward by Jeanquartier, Melrose)**.** *Let $\omega\in\Omega_c^n([0,1]^n)$ be a smooth differential form of top degree on $[0,1]^n$ and $F:(t_1,\ldots,t_n)\in\mathbb{R}_+^n\mapsto t_1^2\ldots t_n^2\in\mathbb{R}$. Then for every $m\in\mathbb{N}$, the map*

$$t\mapsto J(t)=\int_{F^{-1}(t)}\frac{t_n^{-m}\omega}{dF}=\left\langle\delta(t-F),t_n^{-m}\omega\right\rangle$$

*has an asymptotic expansion: $J(t)\sim\sum_{p,q}t^p\log(t)^q a_{p,q}(\omega)$ where $p\in\frac{\mathbb{Z}}{2}$ runs over a finite set of growing arithmetic sequences of rational numbers and $a_{p,q}$ is a* ***distribution*** *supported by the algebraic set $\{F=0\}$.*

*This implies that the map $\varepsilon\longmapsto\int_\varepsilon^1 J(t)dt$ also has an asymptotic expansion: $\int_\varepsilon^1 J(t)dt\sim\sum_{p,q}\varepsilon^p\log(\varepsilon)^q b_{p,q}(\omega)$ where $p\in\frac{\mathbb{Z}}{2}$ runs over a finite set of growing arithmetic sequences of rational numbers and $b_{p,q}$ are distributions supported by $F=\{0\}$.*

*Proof.* The result for smooth forms and real analytic $F$ is due to Jeanquartier [28] [80, Theorem 5.54 p. 155]. Here we need the same result for a polyhomogeneous top form $t_n^{-m}\omega$ and $F=t_1^2\ldots t_n^2$ which is a particular case of the pushforward Theorem of Melrose [38, Thm 3.6 p. 25] [51] by the $b$-map $F$ which yields an index set contained in $\frac{\mathbb{Z}}{2}$ since the $b$-map $F$ vanishes at order 2 on each boundary face of $[0,1]^n$. Let us give a proof based on remarks from Jeanquartier on the Mellin transform [29]. The index set of the asymptotics of $t\mapsto\langle\delta(t-F),t_n^{-m}\varphi\rangle$ is exactly given by the poles with multiplicity of the Mellin transform $\int_0^\infty t^sJ(t)\frac{dt}{t}=\int_{[0,1]^n}F^{s-1}t_n^{-m}\varphi d^nt$ by [29, Prop 4.3 p. 304 and Prop 4.4 p. 306]. By successive Taylor expansion with remainder as follows, start from $\varphi$ then Taylor expand with remainder at order $N$ in $t_1$ keeping other variables $(t_2,\ldots,t_n)$ as parameters, then Taylor expanding successively in $t_2,\ldots,t_n$ with remainder at order $N$ yields: $\varphi(t_1,\ldots,t_n)=\sum_{0\leqslant\alpha_1,\ldots,\alpha_n\leqslant N}\prod t_i^{\alpha_i}c_\alpha$ where $c_\alpha$ depends on $t_i$ iff $\alpha_i=N$. Then plugging under the integral yields that $s \mapsto \int_{[0,1]^n} F^{s-1} t_n^{-m} \varphi d^n t$ has analytic continuation as a meromorphic function on $\mathbb{C}$ with singular terms of the form $\left(\prod_{i=1}^{n-1} \frac{1}{2s+\alpha_i-1}\right) \frac{1}{2s+\alpha_n-1-m}$ hence poles are in $\{s \in \frac{1+m-\mathbb{N}}{2}\}$ with multiplicity at most $n$. □

7.2. **Every $\mathcal{R}\det$ solution of problem 3.2 are obtained by local renormalization.** By Lemma 7.1, $\forall k \in \mathbb{N}$, $(V_1, \dots, V_{k+1}) \mapsto FP|_{\varepsilon=0} Tr_{L^2}\left(e^{-2\varepsilon\Delta}\Delta^{-1}V_1 \dots e^{-2\varepsilon\Delta}\Delta^{-1}V_{k+1}\right)$ is multilinear continuous hence it can be represented as a distributional pairing

$$FP|_{\varepsilon=0} Tr_{L^2}\left(e^{-2\varepsilon\Delta}\Delta^{-1}V_1 \dots e^{-2\varepsilon\Delta}\Delta^{-1}V_{k+1}\right) = \langle \mathcal{R}t_{k+1}, V_1 \boxtimes \dots \boxtimes V_{k+1}\rangle$$

by the multilinear Schwartz kernel Theorem. Exactly as in the proof of subsubsection 4.7.1, we find that for $(V_1, \dots, V_{k+1}) \in C^\infty(M, End(E))^{k+1}$ such that $\operatorname{supp}(V_1) \cap \dots \cap \operatorname{supp}(V_{k+1}) = \emptyset$,

$$FP|_{\varepsilon=0} Tr_{L^2}\left(e^{-2\varepsilon\Delta}\Delta^{-1}V_1 \dots e^{-2\varepsilon\Delta}\Delta^{-1}V_{k+1}\right) = Tr_{L^2}\left(\Delta^{-1}V_1 \dots \Delta^{-1}V_{k+1}\right)$$

where the $L^2$ trace on the r.h.s is well–defined since $WF(\Delta^{-1}V_1) \cap \dots \cap WF(\Delta^{-1}V_{k+1}) = \emptyset$. Therefore arguing as in subsubsection 4.7.1 we find that for $n \leqslant \frac{d}{2}$, $\mathcal{R}t_n$ is a distributional extension of $t_n = \mathbf{G}(x_1, x_2) \dots \mathbf{G}(x_n, x_1)$ and for $n > \frac{d}{2}$, the composition $e^{-2\varepsilon\Delta}\Delta^{-1}V_1 \dots e^{-2\varepsilon\Delta}\Delta^{-1}V_{k+1} \in \Psi^{-2k}(M, E)$ hence of trace class [25, Prop B 21] uniformly in $\varepsilon \in (0, 1]$ hence

$$FP|_{\varepsilon=0} Tr_{L^2}\left(e^{-2\varepsilon\Delta}\Delta^{-1}V_1 \dots e^{-2\varepsilon\Delta}\Delta^{-1}V_{k+1}\right) = Tr_{L^2}\left(\Delta^{-1}V_1 \dots \Delta^{-1}V_{k+1}\right)$$

where the r.h.s. is well–defined as in the case with zeta regularization.

Now let $P_{n,\varepsilon} \in \mathcal{O}_{loc} \otimes \mathbb{C}[\varepsilon^{-\frac{1}{2}}, \log(\varepsilon)]$ from Lemma 7.1 s.t.

$$\lim_{\varepsilon \to 0} Tr_{L^2}\left((e^{-2\varepsilon\Delta}\Delta^{-1}V)^n\right) - P_{n,\varepsilon}(V) = FP|_{\varepsilon=0} Tr_{L^2}\left((e^{-2\varepsilon\Delta}\Delta^{-1}V)^n\right).$$

One should think of $P_{n,\varepsilon}$ as being the **singular part** of $Tr_{L^2}\left(e^{-2\varepsilon\Delta}\Delta^{-1}V_1 \dots e^{-2\varepsilon\Delta}\Delta^{-1}V_n\right)$. Then set $P_\varepsilon(V) = \sum_{n=1}^{\frac{d}{2}} P_{n,\varepsilon}(V)$, we have

$$\begin{aligned} \det_F\left(Id + e^{-2\varepsilon\Delta}\Delta^{-1}V\right) e^{-P_\varepsilon(V)} &= \underbrace{\exp\left(\sum_{n=1}^{\frac{d}{2}} Tr_{L^2}\left((e^{-2\varepsilon\Delta}\Delta^{-1}V)^n\right) - P_{n,\varepsilon}(V)\right)} \\ &\times \underbrace{\det_{[\frac{d}{2}]+1}\left(Id + e^{-2\varepsilon\Delta}\Delta^{-1}V\right)} \end{aligned}$$

by the factorization properties of Gohberg–Krein's determinant [71, d) Thm 9.2 p. 75]. The individual factors underbraced converge as follows:

- $\lim_{\varepsilon\to 0^+} \det_{[\frac{d}{2}]+1}\left(Id + e^{-2\varepsilon\Delta}\Delta^{-1}V\right) = \det_{[\frac{d}{2}]+1}\left(Id + \Delta^{-1}V\right)$ because $e^{-2\varepsilon\Delta}\Delta^{-1}V \to \Delta^{-1}V \in \Psi^{-2}(M, E)$ hence in the Schatten ideal $\mathcal{I}_{[\frac{d}{2}]+1}$ and Gohberg–Krein's determinant $H \mapsto \det_{[\frac{d}{2}]+1}(Id + H)$ depends continuously on $H \in \mathcal{I}_{[\frac{d}{2}]+1}$.
- $\lim_{\varepsilon\to 0^+} \exp\left(\sum_{n=1}^{\frac{d}{2}} Tr_{L^2}\left((e^{-2\varepsilon\Delta}\Delta^{-1}V)^n\right) - P_{n,\varepsilon}(V)\right) = \exp\left(\sum_{n=1}^{\frac{d}{2}} \langle \mathcal{R}t_n, V^{\boxtimes n}\rangle\right)$ where $\mathcal{R}t_n$ is a distributional extension of $t_n = \mathbf{G}(x_1, x_2) \dots \mathbf{G}(x_n, x_1)$ by construction.

Thus it is immediate that

$$\mathcal{R}\det(\Delta+V) = \lim_{\varepsilon\to0^+} \det_F\left(Id + e^{-2\varepsilon\Delta}\Delta^{-1}V\right)e^{-P_\varepsilon(V)} = \exp\left(\sum_{n=1}^{\frac{d}{2}}\left\langle \mathcal{R}t_n, V^{\boxtimes n}\right\rangle\right)\det_{[\frac{d}{2}]+1}\left(Id+\Delta^{-1}V\right)$$

hence it satisfies the representation formula 4.4 which makes it a solution of problem 3.2. If we are given any other solution $\mathcal{R}_2\det$ of problem 3.2, then by the free transitive action of $\mathcal{O}_{loc,[\frac{d}{2}]}$, we know that there exists $Q\in\mathcal{O}_{loc,[\frac{d}{2}]}$ s.t. $\mathcal{R}_2\det(\Delta+V) = e^{Q(V)}\mathcal{R}\det(\Delta+V) = \lim_{\varepsilon\to0^+}\det_F\left(Id+e^{-2\varepsilon\Delta}\Delta^{-1}V\right)e^{(Q-P_\varepsilon)(V)}$ which shows that $\mathcal{R}_2\det$ is obtained by renormalization by subtraction of local counterterms.

## 8. Relation with Gaussian Free Fields.

In the bosonic case, there is a nice interpretation of the renormalized determinants from Theorem 3 in terms of the Gaussian Free Field.

8.0.1. *Probabilistic representation.* We next briefly recall some probabilistic definition of the Gaussian Free Field (GFF) associated to our positive elliptic operator $\Delta$ which is represented as a random distribution on $M$.

**Definition 8.1** (Bundle valued Gaussian Free Field)**.** *Under the geometric assumption from definition 2.6, if $\Delta : C^\infty(E)\mapsto C^\infty(E)$ is **positive, self-adjoint** then the Gaussian free field $\phi$ associated to $\Delta$ is defined as follows: denote by $(e_\lambda)_{\lambda\in\sigma(\Delta)}$ the spectral resolution associated to $\Delta$. Consider a sequence $(c_\lambda)_{\lambda\in\sigma(\Delta)}, c_\lambda\in\mathcal{N}(0,1)$ of independent, identically distributed Gaussian random variables. Then we define the quantum field $\phi$ as the random series*

$$\phi = \sum_{\lambda\in\sigma(\Delta)}\frac{c_\lambda}{\sqrt{\lambda}}e_\lambda \tag{8.1}$$

*where the sum runs over the eigenvalues of $\Delta$ and the series converges almost surely as distributional section in $\mathcal{D}'(M,E)$.*

*The covariance of the Gaussian free field defined above is the Green function:*

$$\boxed{\mathbf{G}(x,y) = \sum_{\lambda\in\sigma(\Delta)}\frac{1}{\lambda}e_\lambda(x)\boxtimes e_\lambda(y)}$$

*where the above series converges in $\mathcal{D}'(M\times M, E\boxtimes E)$.*

A classical result characterizes the support of the functional measure:

**Lemma 8.2** (Regularity of bundle GFF)**.** *Using the notations of definition 8.1, the random section $\phi$ converges almost surely in the Sobolev space $H^s(E)$ for every $s<1-\frac{d}{2}$.*

In Euclidean quantum field theory, there is an analogy between considering a discrete GFF on a lattice with spacing $\sqrt{\varepsilon}$, whose propagator is a discrete Green function which is the inverse of the discrete Laplacian and considering the heat regularized GFF $\phi_\varepsilon = e^{-\varepsilon\Delta}\phi$ whose covariance reads $e^{-2\varepsilon\Delta}\Delta^{-1}$. For discrete Laplacians $\Delta_\varepsilon$ on a regular lattice of mesh $\varepsilon$, there are beautiful results on the asymptotics of $\det(\Delta_\varepsilon)$ [11] (see [46] for related results):

**Theorem 6.** *On the flat torus $\mathbb{T}^2$, for discrete Laplacian $\Delta_\varepsilon$ with mesh $\varepsilon$ and denote by $\phi_\varepsilon$ the corresponding **discrete GFF**, if $V\in C^\infty(\mathbb{T}^2)$ s.t. $\int_{\mathbb{T}^2}V=0$ then:*

$$\frac{\det_\zeta\left(\Delta+V\right)}{\det_\zeta\left(\Delta\right)}=\lim_{\varepsilon\to 0}\frac{\det(\Delta_\varepsilon+V)}{\det(\Delta_\varepsilon)}=\lim_{\varepsilon\to 0^+}\mathbb{E}\left(e^{-\frac{1}{2}\int_{\mathbb{T}^2}V\phi_\varepsilon^2}\right)^{-2}. \tag{8.2}$$

In the bosonic case, replacing lattice regularization by the heat regularized GFF, we prove an analog of the above Theorem and describe all renormalized determinants from Theorem 3 as coming from the local renormalization of Gaussian free fields partition function as follows:

**Theorem 7** (GFF representation)**.** *Under the assumptions of definition 8.1. Let $\phi$ be the Gaussian free field with covariance $\mathbf{G}$. Denote by $\phi_\varepsilon=e^{-\varepsilon\Delta}\phi$ the heat regularized GFF.*

*Then a function $V\mapsto\mathcal{R}\det\left(\Delta+V\right)$ is a renormalized determinant in the sense of definition 3.2 if and only if there exists a sequence $\left(\Lambda_\varepsilon:C^\infty(E)\mapsto C^\infty\left(E\right)\right)_{\varepsilon\in(0,1]}$ of **smooth local** polynomial functionals of minimal degree such that the following limit exists:*

$$\mathcal{R}\det\left(Id+\Delta^{-1}V\right)^{-\frac{1}{2}}=\lim_{\varepsilon\to 0^+}\mathbb{E}\left(\exp\left(-\frac{1}{2}\int_M\langle\phi_\varepsilon,V\phi_\varepsilon\rangle-\Lambda_\varepsilon\left(V\right)(x)dv(x)\right)\right). \tag{8.3}$$

*Furthermore, if $V\in C^\infty(End(E))$ defines a positive operator on $L^2(E)$, we denote by $\mu$ the Gaussian measure of covariance $\Delta^{-1}$ then the limit of measures*

$$\nu=\lim_{\varepsilon\to 0^+}\frac{\exp\left(-\frac{1}{2}\int_M\langle\phi_\varepsilon,V\phi_\varepsilon\rangle-\Lambda_\varepsilon\left(V\right)(x)dv(x)\right)}{\mathbb{E}\left(\exp\left(-\frac{1}{2}\int_M\langle\phi_\varepsilon,V\phi_\varepsilon\rangle-\Lambda_\varepsilon\left(V\right)(x)dv(x)\right)\right)}\mu \tag{8.4}$$

*exists as a Gaussian measure on $\mathcal{D}'\left(M\right)$ with covariance $\left(\Delta+V\right)^{-1}$ and $\nu$ is absolutely continuous w.r.t. $\mu$ iff $1\leqslant d\leqslant 3$ otherwise the measures $(\nu,\mu)$ are mutually singular.*

The intuitive idea is very simple. In QFT the renormalization problem arises from the fact that fields are irregular distributions then a natural idea is to study a regularized version of the field and see if one can perform an explicit renormalization of the partition function by subtracting **explicit local counterterms** in the action functional. The first part of Theorem 7 follows from Theorem 3 once we reformulate the partition function $\mathbb{E}(e^{-\int_M\langle\varphi_\varepsilon,V\varphi_\varepsilon\rangle})$, where $\phi_\varepsilon=e^{-\varepsilon\Delta}\phi$ is the smeared GFF, in terms of Fredholm determinants $\det_F\left(Id+\Delta^{-1}e^{-\varepsilon\Delta}Ve^{-\varepsilon\Delta}\right)$ which is the goal of the next paragraph. In a companion paper [14, Prop 1.4], we give a simple derivation of the above Theorem 7 using elementary commutator arguments when $\dim(M)\leqslant 4$.

8.0.2. *Fredholm determinants and partition functions.* The following Lemma relates partition functions and Fredholm determinants:

**Lemma 8.3** (Field regularization.)**.** *Under the assumptions of definition 8.1, let $\phi_\varepsilon=e^{-\varepsilon\Delta}\phi$ be the mollified GFF.*

*Then for every $\varepsilon>0$, the following relation holds true:*

$$\mathbb{E}\left(\exp\left(-\frac{1}{2}\int_M\langle\phi_\varepsilon,V\phi_\varepsilon\rangle\,dv(x)\right)\right)=det_F\left(Id+e^{-\varepsilon\Delta}\Delta^{-1}e^{-\varepsilon\Delta}V\right)^{-\frac{1}{2}}.$$

*Proof.* This is an immediate consequence of [35, Remark 1 p. 211] which allows to write $\mathbb{E}\left(\exp\left(-\frac{1}{2}\int_M \langle\phi_\varepsilon, V\phi_\varepsilon\rangle dv(x)\right)\right) = \exp\left(-\frac{1}{2}Tr_{L^2}\left(Id+\widehat{V}_\varepsilon\right)\right)$ for $\|V\|_\infty$ small enough, where

$$\widehat{V}_\varepsilon = e^{-\varepsilon\Delta}\Delta^{-\frac{1}{2}}V\Delta^{-\frac{1}{2}}e^{-\varepsilon\Delta}$$

is positive, self-adjoint and smoothing hence trace class on $L^2(E)$. We can expand the term $Tr_{L^2}\log\left(Id+\widehat{V}_\varepsilon\right)$ in power series and use the cyclicity of the $L^2$ trace to identify $\exp\left(-\frac{1}{2}Tr_{L^2}\left(Id+\widehat{V}_\varepsilon\right)\right)$ with the power series defining the Fredholm determinant $\det_F\left(Id+e^{-\varepsilon\Delta}\Delta^{-1}e^{-\varepsilon\Delta}V\right)^{-\frac{1}{2}}$. A very similar proof can be found in [14, subsubsections 3.0.1 and 3.0.2] where we relate the Wick renormalized partition function with the Gohberg–Krein determinant $\det_2$. □

8.1. **The renormalized functional measure.** In the previous part, we have constructed renormalized functional determinants to rigorously define the partition function. The following Proposition proves the second part of Theorem 7 and answers some natural questions about the corresponding renormalized functional measure.

**Proposition 8.4.** *Under the assumptions of definition 8.1, assume $V\in C^\infty(End(E))$ is Hermitian. Let $\mu$ denote the GFF measure on $\mathcal{D}'(M,E)$ with covariance $\mathbf{G}$ which is the Schwartz kernel of $\Delta^{-1}$. Then there exists $P_\varepsilon(.) = \int_M \Lambda_\varepsilon(.) \in \mathcal{O}_{loc,[\frac{d}{2}]}\left(J^{d-3}E\right)\otimes_{\mathbb{C}}\mathbb{C}[\varepsilon^{-\frac{1}{2}},\log(\varepsilon)]$ s.t. the limit*

$$\nu = \lim_{\varepsilon\to 0^+}\exp\left(-\frac{1}{2}\int_M\left(\langle\phi_\varepsilon, V\phi_\varepsilon\rangle - \Lambda_\varepsilon(V)\right)dv(x)\right)\mu$$

*converges to a Gaussian measure on $\mathcal{D}'(M,E)$ which is* ***absolutely continuous*** *w.r.t. $\mu$ if $d=(2,3)$ and the measure $(\mu,\nu)$* ***are mutually singular*** *when $d\geqslant 4$.*

$\Lambda$ depends on the $(d-3)$-jet of $V$ in the above proposition.

*Proof.* Define $\nu_\varepsilon = \frac{\exp\left(-\frac{1}{2}\int_M(\langle\phi_\varepsilon,V\phi_\varepsilon\rangle-\Lambda_\varepsilon(V))dv(x)\right)}{\mathbb{E}\left(\exp\left(-\frac{1}{2}\int_M(\langle\phi_\varepsilon,V\phi_\varepsilon\rangle-\Lambda_\varepsilon(V))dv(x)\right)\right)}\mu$ for $\varepsilon>0$. This is a Gaussian measure whose covariance is $\left(\Delta+e^{-\varepsilon\Delta}Ve^{-\varepsilon\Delta}\right)^{-1}$ by [35, Prop 9.3.2 p. 213]. When $\varepsilon\to 0^+$, this covariance converges to $(\Delta+V)^{-1}$ as **bilinear forms on** $C^\infty(M)\times C^\infty(M)$ for the weak topology [35, iv) p. 208] since $e^{-\varepsilon\Delta}\to Id$ in the **strong operator topology** when $\varepsilon\to 0^+$. A necessary and sufficient condition for the renormalized measure to be absolutely continuous w.r.t. the initial measure is given by a Theorem of Shale [72, Thm I.23 p. 41] is that $\Delta^{-\frac{1}{2}}V\Delta^{-\frac{1}{2}}\in\Psi^{-2}(M,E)$ is Hilbert–Schmidt which holds true only if $\dim(M)=d\leqslant 3$. □

## 9. Quillen's determinant line bundle.

We recall the definition of Quillen's determinant line bundle which is an adaptation of the definition of Segal [70, p. 137-138], Furutani [32] and Melrose–Rochon [52] where holomorphicity properties are manifest. The reader can also look at [68, section 5.3 p. 642] for a very nice account of determinant line bundles for families of Ψdos.

**Definition 9.1** (Quillen's universal determinant line bundle)**.** *Using the notations of subsubsection 1.0.3. Recall $\mathcal{I}_1(\mathcal{H})$ denotes the ideal of trace class operators on some Hilbert space $\mathcal{H}$. Let $(T_b)_{b\in B}$ be a holomorphic family of Fredholm operators from $\mathcal{H}_0 \mapsto \mathcal{H}_1$ of index 0, parametrized by a complex Banach manifold $B$. Consider the bundle*

$$\mathcal{G} = \bigcup_{b\in B} T_b(Id + \mathcal{I}_1(\mathcal{H}_1)) \simeq B \times (Id + \mathcal{I}_1(\mathcal{H}_1))$$

*which fibers over the complex Banach manifold $B$.*

*Then we define the determinant line bundle $\boldsymbol{Det} \mapsto B$ to be the quotient $\mathcal{G}\times\mathbb{C}/\sim$ where $(A(Id+T), z) \sim (A, \det_F(Id+T)z)$. The canonical section $\underline{\det}(T)$ is defined to be the equivalence class $T \mapsto [T, 1]$.*

This definition is functorial since it works for any holomorphic family $(T_b)_{b\in B}$ and holomorphicity is checked as in the work of Furutani [32]. Quillen's line bundle is recovered by letting $B$ to be the space $\mathbf{Fred}_0(\mathcal{H}_0, \mathcal{H}_1)$ of Fredholm operators of index 0 as proved by Furutani [32, section 2 and prop 2.1]. Let us recall that

**Lemma 9.2.** *The canonical section $T \mapsto \underline{\det}(T) = [T, 1]$ vanishes if and only if $T$ is non-invertible.*

*Proof.* $[T, 1] \simeq [\tilde{T}, 0]$ means there exists $Id+A, A \in \mathcal{I}^1$ s.t. $\tilde{T}(Id+A) = T$ and $\det_F(Id+A) = 0$ hence $Id + A$ is non-invertible and so is $T$. Conversely, even if $T$ is non-invertible, there is a finite rank operator $t$ such that $T + t$ invertible since $T$ is Fredholm of index 0. Therefore $T = (T + t)(Id - (T + t)^{-1}t)$ where $(Id - (T + t)^{-1}t)$ is in the determinant class and is non-invertible. Finally $[T, 1] \sim [T + t, \det_F((Id - (T + t)^{-1}t))] = 0$. □

## 10. Proof of Theorem 4.

We follow the notations from subsubsection 1.0.3. The way Quillen trivializes the line bundle is by constructing a smooth Hermitian metric on $\mathcal{L}$ named Quillen's metric and he calculates explicitely the curvature of the corresponding Chern connection which is exactly the Kähler form on $\mathcal{A}$. Then he shows that by modifying the Hermitian metric, one can produce a modified Chern connection $\nabla$ which is flat. It follows from the contractibility of $\mathcal{A}$ that flat sections for $\nabla$ trivialize $\mathcal{L}$ **holomorphically**. Here the setting is slightly different. Our approach to holomorphic trivialization is more direct and does not use Quillen metrics. We already know that the canonical section $\iota^*\underline{\det}$ has the same zeros on $\mathcal{A}$ as any solution $\mathcal{R}\det$ of Theorem 3. Hence, we expect that the ratio $\frac{\iota^*\underline{\det}(D)}{\mathcal{R}\det(D)}$ is holomorphic without zeros on $\mathcal{A}$. It remains to show that this is well–defined and locally bounded in order to conclude that the ratio is a holomorphic section without zeros by proposition 11.5, hence it yields a holomorphic trivialization of $\mathcal{L}$.

Following Segal and Furutani, we define open sets $U_t \subset \mathbf{Fred}_0(\mathcal{H})$ indexed by finite rank operators $t$ such that $U_t = \{T \in \mathbf{Fred}_0(\mathcal{H}) \text{ s.t. } T+t \text{ invertible}\}$. Since elements in $\mathbf{Fred}_0(\mathcal{H})$ have Fredholm index 0, the collection $(U_t)_t$ forms an open cover of $\mathbf{Fred}_0(\mathcal{H})$. Then we trivialize $\mathcal{L}$ over $U_t$ by the never vanishing section $T \in U_t \mapsto [T+t, 1]$ which is holomorphic by the proof of Furutani. In the local trivialization, the canonical section

$$T \mapsto \underline{\det(T)} = [T, 1]$$

is identified with the holomorphic function $\det_F(Id-(T+t)^{-1}t)$ since $[T,1] \sim [T+t, \det_F(Id-(T+t)^{-1}t)] = \det_F(Id-(T+t)^{-1}t)[T+t,1]$.

Now we shall prove a technical

**Lemma 10.1.** *Let $T_0$ be an invertible operator in $\iota(\mathcal{A})$ such that for all $T \in \iota(\mathcal{A})$, $T - T_0$ is in the Schatten ideal $\mathcal{I}_{[\frac{d}{k}]+1}$, $k = (1,2)$.*

*It follows that for $p = [\frac{d}{k}]+1$, Gohberg–Krein's determinant $\det_p\left(Id + T_0^{-1}(T-T_0)\right)$ is holomorphic on $\iota(\mathcal{A})$. Then the section*

$$T \in \iota(\mathcal{A}) \longmapsto det_p\left(Id + T_0^{-1}(T-T_0)\right)^{-1}\underline{\det(T)} \tag{10.1}$$

*defines a global holomorphic section of* **Det** $\mapsto \iota(\mathcal{A})$ *which never vanishes.*

*Proof.* It suffices to prove the claim on each open subset $U_t \cap \iota(\mathcal{A})$ where the canonical section $T \mapsto \underline{\det(T)}$ is identified with $T \in U_t \mapsto \det_F(Id-(T+t)^{-1}t)$ by the local trivialization.

Use the identity $Id + T_0^{-1}(T-T_0) = T_0^{-1}T$ and $Id - (T+t)^{-1}t = (T+t)^{-1}T$. By the multiplicativity of Fredholm determinants, for every invertible $T \in U_t \cap \iota(\mathcal{A})$, we find that [19]

$$\begin{aligned}
& \det_F(Id-(T+t)^{-1}t)\det_p\left(Id + T_0^{-1}(T-T_0)\right)^{-1} \\
= \; & \det_F(Id-(T+t)^{-1}t)\det_F\left(Id + R_p\left(T_0^{-1}(T-T_0)\right)\right)^{-1} \\
= \; & \det_F\left((T+t)^{-1}T_0(Id+T_0^{-1}(T-T_0))(Id+R_p(T_0^{-1}(T-T_0)))^{-1}\right).
\end{aligned}$$

For every $T \in U_t \cap \iota(\mathcal{A})$, the operator $(T+t)^{-1}T_0$ is invertible. For such $T$, we observe by definition of $R_p$ that

$$\begin{aligned}
& (Id+T_0^{-1}(T-T_0))(Id+R_p(T_0^{-1}(T-T_0)))^{-1} \\
= \; & (Id+T_0^{-1}(T-T_0))(e^{\sum_{k=1}^{p-1}\frac{(-1)^{k+1}}{k}(T_0^{-1}(T-T_0))^k}(Id+T_0^{-1}(T-T_0)))^{-1} \\
= \; & e^{\sum_{k=1}^{p-1}\frac{(-1)^{k}}{k}(T_0^{-1}(T-T_0))^k}
\end{aligned}$$

where the term $e^{\sum_{k=1}^{p-1}\frac{(-1)^{k}}{k}(T_0^{-1}(T-T_0))^k}$ is well–defined thanks to the holomorphic functional calculus for the compact operator $T_0^{-1}(T-T_0)$ and is easily seen to be invertible by the spectral mapping theorem for holomorphic functions of bounded operators. Indeed, for every bounded operator $A$, $f : \Omega \subset \mathbb{C} \mapsto \mathbb{C}$ holomorphic in some neighborhood $\Omega$ of $\sigma(A)$, $f(A)$ is well–defined with $\sigma(f(A)) = f(\sigma(A))$ by the spectral mapping Theorem [44, Thm 2.3.6 p. 22]. In our case, this gives that 0 is not in the spectrum of $e^{\sum_{k=1}^{p-1}\frac{(-1)^{k}}{k}(T_0^{-1}(T-T_0))^k}$. Finally

$$(T+t)^{-1}T_0(Id+T_0^{-1}(T-T_0))(Id+R_p(T_0^{-1}(T-T_0)))^{-1} = (T+t)^{-1}T_0e^{\sum_{k=1}^{p-1}\frac{(-1)^{k}}{k}(T_0^{-1}(T-T_0))^k}$$

[19] For every trace class $H$, we are using the fact that $(Id+H)^{-1} \in Id + \mathcal{I}_1$ and $\det_F((Id+H)^{-1}) = \det_F(Id+H)^{-1}$.

is invertible for every $T \in U_t \cap \iota(\mathcal{A})$ and is the composition of two operators of the form $Id + \mathcal{I}_1$ and $(Id + \mathcal{I}_1)^{-1}$ hence it belongs to the determinant class. Therefore, its Fredholm determinant never vanishes. It follows that $T \in U_t \cap \iota(\mathcal{A}) \cap \text{invertible} \mapsto \det_F(Id - (T + t)^{-1}t)\det_p\left(Id + T_0^{-1}(T - T_0)\right)^{-1}$ extends uniquely as a never vanishing holomorphic function on $U_t \cap \iota(\mathcal{A})$. □

Lemma 10.1 says the ratio $P+\mathcal{V} \in \mathcal{A} \mapsto \det_{[\frac{d}{k}]+1}(Id+P^{-1}\mathcal{V})^{-1}\iota^*\underline{\det}(P+\mathcal{V})$ never vanishes over $\mathcal{A}$. Furthermore Corollary 3.3 states that $\mathcal{R}\det(P+\mathcal{V}) = \exp(g(\mathcal{V}))\det_{[\frac{d}{k}]+1}(Id+P^{-1}\mathcal{V})$ where $g$ is a polynomial function, therefore $\exp(g(\mathcal{V}))$ never vanishes and the holomorphic section $\sigma : P + \mathcal{V} \in \mathcal{A} \mapsto \mathcal{R}\det(P + \mathcal{V})^{-1}\iota^*\underline{\det}(P + \mathcal{V})$ never vanishes over $\mathcal{A}$ and defines a holomorphic trivialization of $\mathcal{L}$: $\tau : \mathcal{O}(\mathcal{L}) \mapsto Hol(\mathcal{A})$ such that the canonical section $\iota^*\underline{\det}(T)$ is sent to the entire function $T \in \mathcal{A} \mapsto \mathcal{R}\det(T)$. The second claim follows from the action of the renormalization group as in Theorem 3. Finally, every non vanishing section $\sigma$ defines canonically a flat connection $\nabla$ whose flat section is $\sigma$.

## 11. Appendix.

**11.1. Wave front set of Schwartz kernels of local polynomial functionals.** We give the proof of the following

**Lemma 11.1.** *Let $P$ be a continuous polynomial function on $C^\infty(M)$ such that $P$ is local in the sense*

$$D^2P(w; u, v) = 0 \tag{11.1}$$

*when $(u, v)$ have disjoint supports and the linear term of $P$ is given by integration against a smooth function. If $WF\left([\mathbf{D^2P(V)}]\right) \subset N^*\left(d_2 \subset M^2\right)$ for all $V \in C^\infty(M)$ then $P \in \mathcal{O}_{loc}(C^\infty(M))$.*

*Proof.* Equation 11.1 implies that all Gâteaux differentials $D^nP(0)$ of $P$ at $0 \in C^\infty(M)$ have their Schwartz kernels $[\mathbf{D^nP}(0)] \in \mathcal{D}'(M^n)$ supported on the deepest diagonal $d_n \subset M^n$ by [7, Proposition V.5] and that $P$ is **additive** in the sense of [7]. Since $P$ is a polynomial function, it equals its Taylor expansion $P(V) = \sum_{n=1}^{\deg(P)} P_n(V)$ where $P_n$ homogeneous of degree $n$.

The smoothness condition on the linear term in $P$ together with the microlocal condition on $[\mathbf{D^2P(V)}] \in \mathcal{D}'(M \times M), \forall V \in C^\infty(M)$ imply that $DP(0)$ is represented by integration against smooth function $[\mathbf{DP(0)}] \in C^\infty(M)$.

Therefore by uniqueness of the Taylor expansion each $P_n$ satisfies equation 11.1. Let $\tilde{P}_n$ be the multilinear map corresponding to $P_n$ and its Schwartz kernel $[\mathbf{P_n}] \in \mathcal{D}'(M^n)$ whose existence is given by the kernel Theorem [7]. The Schwartz kernel $[\mathbf{P_n}] \in \mathcal{D}'(M^n)$ is a distribution carried by the deepest diagonal by locality of $P_n$. By a Theorem of Laurent Schwartz, $[\mathbf{P_n}]$ has an expression in local coordinates $(x_1, \dots, x_n)$ in $U^n$ as

$$[\mathbf{P_n}](x_1, \dots, x_n) = \sum_{[\alpha]} f_{[\alpha]}(x_1)\partial_{x_2}^{\alpha_2} \dots \partial_{x_n}^{\alpha_n}\partial_{x_1}^{\alpha_1}\delta_{\{0\}}^{\mathbb{R}^{d(n-1)}}(x_1 - x_2, \dots, x_1 - x_n)$$

where the sum over the multiindices $[\alpha]=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^{dn}$ is finite and $f_{[\alpha]}$ is a distribution in the variable $x_1$. It follows that the Schwartz kernel of the second Gâteaux differential has the representation in local coordinates

$$[\mathbf{D^2P(V)}](x,y)=\sum_{[\alpha]} f_{[\alpha]}(x)\partial_x^{\alpha_2}V(x)\dots\partial_x^{\alpha_n}V(x)\partial_y^{\alpha_1}\delta_{\{0\}}^{\mathbb{R}^d}(x-y)$$

which implies $P$ satisfies condition 2 of [7, Lemma VI.9]. By [7, Lemma VI.9], this means $V\in C^\infty(M)\mapsto[\mathbf{DP(V)}]\in C^\infty(M)$ is smooth. To summarize, $P$ is additive, its differential $DP(V)$ is represented by integration against a smooth function $[\mathbf{DP(V)}]\in C^\infty(M)$ and $V\in C^\infty(M)\mapsto[\mathbf{DP(V)}]\in C^\infty(M)$ is smooth hence by [7, Theorem I.2], $P\in\mathcal{O}_{loc}(C^\infty(M))$. $\square$

11.2. **Sharpness of the bound from the main Theorem.** We give an application of the Hadamard Theorem 1 by giving an example where the bound from Theorem 3 on the order of the entire function $z\mapsto\mathcal{R}\det(P+z\mathcal{V})$ is sharp.

**Lemma 11.2.** *Let $\Delta$ be the Laplace–Beltrami operator of some Riemannian manifolds $(M,g)$ of dimension $d$. For any entire function $f:\mathbb{C}\mapsto\mathbb{C}$ s.t. $f(z)=0\Leftrightarrow\ker(\Delta+z)\neq\{0\}$ with multiplicity $\dim(\ker(\Delta+z))$, we must have the order $\rho(f)\geqslant[\frac{d}{2}]+1$.*

This proves the bound from problem 3.2 is optimal.

*Proof.* Note that $f(z)=0\implies -z\in\sigma(\Delta)$. By Weyl's law for spectral functions of positive, elliptic pseudodifferential operators [33, Thm 2.1 p. 825], the number of eigenvalues $n_L(\Delta)$ of $\Delta$ less than $L$ grows like a symplectic volume $\int_{\{\sigma(\Delta)(x;\xi)\leqslant L\}\subset T^*M} d^dxd^d\xi\sim_{L\to+\infty} CL^{\frac{d}{2}}$. This implies for $p=[\frac{d}{2}]+1$ that $Tr_{L^2}(\Delta^{-p})=\sum_{z\in\{f=0\}}|z|^{-p}<+\infty$ and $Tr_{L^2}\left(\Delta^{1-p}\right)=\sum_{z\in\{f=0\}}|z|^{-p+1}=+\infty$ hence $\rho(f)\geqslant[\frac{d}{2}]+1$ by Theorem 1. $\square$

Both results show that the solution to the problem of finding entire functions with prescribed zeros is **not unique**, the non unicity is due to the critical exponents of zeros which forces the entire function to have non zero order. So there is an ambiguity relating all possible solutions of the problem which is of the form exp(Polynomial) by Hadamard's factorization Theorem.

11.2.1. *Proof of Lemma 4.4.* The composite operator $\Delta^{-\frac{1}{4}}V\Delta^{-\frac{1}{4}}$ is a pseudodifferential of order 0 in $\Psi^0(M,E)$ by the composition Theorem. Therefore by the Calderon Vaillancourt Theorem, we can choose $V\in\mathrm{Diff}^1(M,E)$ in some small enough neighborhood $\mathcal{U}$ of 0 so that $\max\left(\|\Delta^{-\frac{1}{4}}V\Delta^{-\frac{1}{4}}\|_{\mathcal{B}(H^{\frac{1}{2}},H^{\frac{1}{2}})},\|\Delta^{-\frac{1}{4}}V^*\Delta^{-\frac{1}{4}}\|_{\mathcal{B}(H^{\frac{1}{2}},H^{\frac{1}{2}})}\right)\leqslant\frac{\sqrt{\delta}}{4}$. This yields for every $z$:

$$\begin{aligned}
&Re\left\langle u,(\Delta+V-z)\,u\right\rangle = Re\left\langle\Delta^{\frac{1}{4}}u,\left(\Delta^{\frac{1}{2}}+\Delta^{-\frac{1}{4}}V\Delta^{-\frac{1}{4}}-z\Delta^{-\frac{1}{2}}\right)\Delta^{\frac{1}{4}}u\right\rangle\\
\geqslant\ & C\left(\sqrt{\delta}\|u\|^2_{H^{\frac{1}{2}}}-\|\Delta^{-\frac{1}{4}}V\Delta^{-\frac{1}{4}}\|_{\mathcal{B}(H^{\frac{1}{2}},H^{\frac{1}{2}})}\|u\|^2_{H^{\frac{1}{2}}}-Re(z)\delta^{-\frac{1}{2}}\|u\|^2_{H^{\frac{1}{2}}}\right)\\
\geqslant\ & C\left(\sqrt{\delta}-\|\Delta^{-\frac{1}{4}}V\Delta^{-\frac{1}{4}}\|_{\mathcal{B}(H^{\frac{1}{2}},H^{\frac{1}{2}})}-\frac{Re(z)}{\sqrt{\delta}}\right)\|u\|^2_{H^{\frac{1}{2}}},
\end{aligned}$$

where $C$ is some constant such that $C\|u\|_{H^{\frac{1}{2}}} \leqslant \|\Delta^{\frac{1}{4}}u\|_{L^2} \leqslant C^{-1}\|u\|_{H^{\frac{1}{2}}}$. Hence when $Re(z) \leqslant \frac{\delta}{2}$: $Re\left\langle u, (\Delta + V - z)\, u\right\rangle \geqslant C\frac{\sqrt{\delta}}{4}\|u\|^2_{L^2}$. We have a similar estimate for the adjoint $V^*$ which implies that $\{Re(z) \leqslant \frac{\delta}{2}\}$ lies in the resolvent set of $(\Delta + V + z)$.

11.3. **Proof of Proposition 4.9.** Recall $(\Delta + B)_\pi^{-s}$ is the complex power defined in terms of the spectral cut at angle $\pi$. The first formula we need to establish:

$$(\Delta + B)_\pi^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t(\Delta+B)} t^{s-1} dt$$

is widely used in the mathematical physics litterature to define complex powers of Schrödinger type operators. Since we work with non-self-adjoint operators, we need to justify it. We first define

$$(\Delta + B)^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t(\Delta+B)} t^{s-1} dt$$

where the integral on the r.h.s, which is valued in $\mathcal{B}(H^s, H^s)$, converges since on $\int_1^\infty$ we use the exponential decay of the semigroup and on $\int_0^1$, it is well defined for $Re(s) > 0$. So we just defined a holomorphic family of operators $((\Delta + B)^{-s})_{Re(s)>0} : C^\infty(M) \mapsto \mathcal{D}'(M)$. To extend it to the complex plane and to make the connection with actual complex powers, we shall identify it with the definition of complex powers using the contour integral and resolvent instead of the Mellin transform of the heat kernel. In Gilkey's book [34], the heat operator $e^{-t(\Delta+B)}$ for non-self-adjoint operators is expressed in terms of the resolvent by the contour integral

$$e^{-t(\Delta+B)} = \frac{i}{2\pi} \int_\gamma e^{-t\lambda} \left(\Delta + B - \lambda\right)^{-1} d\lambda$$

where the contour integral converges for $t > 0$ by the exponential decay of $e^{-t\lambda}$ since the contour $\gamma$, oriented clockwise, is chosen to be some $V$ shaped curve which contains strictly the angular sector $\mathcal{R}$ from Lemma 4.6 and $\gamma$ is contained in the half–plane $Re(\lambda) \geqslant 0$. We saw that the complex power $(\Delta + B)_\pi^{-s}$ is defined using the spectral cut at angle $\pi$:

$$(\Delta + B)_\pi^{-s} = \frac{i}{2\pi} \int_{\tilde{\gamma}} \lambda^{-s} \left(\Delta + B - \lambda\right)^{-1} d\lambda$$

where the operator valued integral converges absolutely for $Re(s) > 1$ in $\mathcal{B}(L^2, L^2)$ and $\tilde{\gamma} = \{re^{i\pi}, \infty > r \geqslant \rho\} \cup \{\rho e^{i\theta}, \theta \in [\pi, -\pi]\} \cup \{re^{-i\pi}, \rho \leqslant r < \infty\}$.

Since the spectrum of $\Delta + B$ is contained in some neighborhood of $[\frac{\delta}{2}, +\infty)$, we can deform the contour $\tilde{\gamma}$ to the contour $\gamma$ used in defining the heat operator without crossing $\sigma(\Delta + B)$. Using the estimates on the resolvent of Lemma 4.6, Cauchy's formula and contour deformation avoiding $\sigma(\Delta + B)$, it is simple to show that

$$\int_{\tilde{\gamma}} \lambda^{-s} \left(\Delta + B - \lambda\right)^{-1} d\lambda = \int_\gamma \lambda^{-s} \left(\Delta + B - \lambda\right)^{-1} d\lambda$$

where both sides converge absolutely for $Re(s) > 1$. Now we use the formula $\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t\lambda} dt$ which makes sense for the branch $\log(\lambda) = \log(|\lambda|) + i \arg(\lambda), -\pi \leqslant$

$\arg(\lambda) \leqslant \pi$ since $Re(\lambda) > 0$ and $Re(s) > 1$. Therefore

$$(\Delta+B)_\pi^{-s} = \frac{i}{2\pi}\int_\gamma \left(\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-t\lambda}dt\right)(\Delta+B-\lambda)^{-1}d\lambda$$
$$= \frac{i}{2\pi}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\left(\int_\gamma e^{-t\lambda}(\Delta+B-\lambda)^{-1}d\lambda\right)dt = \frac{i}{2\pi}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-t(\Delta+B)}dt$$

where we could invert the integrals since everything converges when $Re(s) > 1$. The above discussion also shows that for any differential operator $Q \in \mathrm{Diff}^1(M,E)$, we have

$$(\Delta+B)_\pi^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty e^{-t(\Delta+B)}t^{s-1}dt \in \mathcal{B}(L^2, H^{-1}).$$

This proves that one can define the complex powers with spectral cut using the heat kernel even in this non-self-adjoint setting.

11.3.1. *Taking the trace.* We want to prove the relation

$$Tr_{L^2}\left(Q(\Delta+B)_\pi^{-s}\right) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}Tr_{L^2}\left(Qe^{-t(\Delta+B)}\right)dt$$

which means that we want to take the functional trace on both sides of the previous identity. Start again from the relation $Q(\Delta+B)_\pi^{-s} = \frac{i}{2\pi}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}Qe^{-t(\Delta+B)}dt$. One key idea is that a sufficiently smoothing operator will be trace class and has continuous Schwartz kernel. For such operator, the $L^2$ trace coincides with the flat trace $Tr^\flat$ defined simply by integrating the Schwartz kernel of the operator restricted on the diagonal against a smooth density.

By the work of Seeley, we know that $(\Delta+B)_\pi^{-s} \in \Psi^{-2s}(M)$, then by composition of pseudodifferential operators $Q(\Delta+B)_\pi^{-s} \in \Psi^{-2s+1}(M)$. This implies that as soon as $Re(s) > \frac{d+1}{2}$, $Q(\Delta+B)_\pi^{-s}$ is trace class and the left hand side $Tr_{L^2}\left(Q(\Delta+B)_\pi^{-s}\right)$ is well–defined and $Tr_{L^2}\left(Q(\Delta+B)_\pi^{-s}\right) = \frac{1}{\Gamma(s)}Tr_{L^2}\left(\int_0^\infty t^{s-1}Qe^{-t(\Delta+B)}dt\right)$. To exchange the trace and the integral on the r.h.s, note that $Tr_{L^2}\left(Q(\Delta+B)_\pi^{-s}\right) = Tr^\flat\left(Q(\Delta+B)_\pi^{-s}\right)$ since $Q(\Delta+B)_\pi^{-s}$ is trace class when $Re(s) > \frac{d+\deg(Q)}{2}$ and has continuous kernel arguing as in [59, p. 102–103]. Therefore

$$Tr_{L^2}\left(Q(\Delta+B)_\pi^{-s}\right) = \frac{1}{\Gamma(s)}Tr^\flat\left(\lim_{\varepsilon\to0^+}\lim_{\Lambda\to+\infty}\int_\varepsilon^\Lambda t^{s-1}Qe^{-t(\Delta+B)}\right)dt.$$

However note that for $t$ in $[\varepsilon,\Lambda]$, it is immediate to prove that $t\mapsto Qe^{-t(\Delta+B)}$ is continuous and uniformly bounded in smoothing operators, therefore we can invert the flat traces and the integral to get $Tr^\flat\left(\int_\varepsilon^\Lambda t^{s-1}Qe^{-t(\Delta+B)}\right) = \int_\varepsilon^\Lambda t^{s-1}Tr^\flat\left(Qe^{-t(\Delta+B)}\right) = \int_\varepsilon^\Lambda t^{s-1}Tr_{L^2}\left(Qe^{-t(\Delta+B)}\right)$ since $\left(Qe^{-t(\Delta+B)}\right) \in \Psi^{-\infty}$. To conclude, it suffices to show that under the assumption that $Re(s) > \frac{d+1}{2}$, the integrand $t^{s-1}Tr_{L^2}\left(Qe^{-t(\Delta+B)}\right)$ is Riemann integrable on $(0,+\infty)$. But this follows almost immediately from the bound (using the fact that $Q$ is a differential operator of degree 1) [34, Lemma 1.9.3 p. 77–78],[3, Thm 2.30 p. 87]

$$\forall t\in(0,1], |Tr_{L^2}\left(Qe^{-t(\Delta+B)}\right)| \leqslant Ct^{-\frac{(d+1)}{2}} \implies |t^{s-1}Tr_{L^2}\left(Qe^{-t(\Delta+B)}\right)| \leqslant Ct^{Re(s)-1-\frac{(d+1)}{2}}$$

where the r.h.s. is absolutely integrable near 0 and

$$\forall t \in [1,+\infty), |Tr_{L^2}\left(Qe^{-t(\Delta+B)}\right)| \leqslant \|e^{-(t-\frac{1}{2})(\Delta+B)}\|_{\mathcal{B}(L^2,L^2)}\|Qe^{-\frac{1}{2}(\Delta+B)}\|_{\mathcal{B}(L^2,H^r)} \leqslant Ce^{(t-\frac{1}{2})\frac{\delta}{2}}$$

for any $r > d$, which uses the exponential decay of the semigroup $e^{-t(\Delta+B)}$, the smoothing properties of $Qe^{-\frac{1}{2}(\Delta+B)}$ and allows us to control the integral for large times. Finally once the identity is proved in some domain $Re(s) > \frac{d+1}{2}$, the analytic continuation takes care of extending the relation on the whole complex plane.

11.3.2. *Proof of Lemma 4.15.*

*Proof.* For every real number $s$, a symbol $p \in S^s_{1,0}(\mathbb{R})$ iff $p$ is in $C^\infty(\mathbb{R})$ and $|\partial^j_\xi p(\xi)| \leqslant C_j(1+|\xi|)^{s-j}$ [79, Lemm 1.2 p. 295] for every $j \in \mathbb{N}$. Observe that the function $p_t : \xi \in \mathbb{R} \mapsto e^{-t|\xi|^2}$ defines a family $(p_t)_{t\in[0,+\infty)}$ of symbols in $S^0_{1,0}(\mathbb{R})$ such that $p_t \underset{t\to 0}{\to} 1$ in $S^{+0}_{1,0}(\mathbb{R})$. Indeed, for $k \in \mathbb{N}$ and for $t$ in some compact interval $[0,a], a > 0$, we find by direct computation that: $(1+|\xi|)^k|\partial^k_\xi e^{-t\xi^2}| \leqslant C(1+|\xi|)^k \sum_{0\leqslant l\leqslant \frac{k}{2}} t^{k-l}|\xi|^{k-2l}e^{-t\xi^2}$ where the constant $C$ depends only on $k$.
When $|\xi| \geqslant a$, the function $t \in [0,+\infty) \mapsto (t^{k-l}\xi^{k-2l})e^{-t\xi^2}$ goes to 0 when $t = 0, t \to +\infty$ and reaches its maximum when $\frac{d}{dt}\left((t^{k-l}\xi^{k-2l})e^{-t\xi^2}\right) = ((k-l)t^{k-l-1}\xi^{k-2l} - t^{k-l}\xi^{k-2l+2})e^{-t\xi^2} = ((k-l) - t\xi^2)t^{k-l-1}\xi^{k-2l}e^{-t\xi^2} = 0$ for $t = \frac{k-l}{\xi^2}$. Hence when $|\xi| \geqslant a$,

$$\sup_{t\in[0,a]} (1+|\xi|)^k|(t^{k-l}\xi^{k-2l})|e^{-t\xi^2} \leqslant (k-l)^{k-l}(1+|\xi|)^k|\xi|^{-k} \leqslant (k-l)^{k-l}(1+a^{-k})^k.$$

On the other hand, if $|\xi| \leqslant a, t \in [0,a]$, we find that $(1+|\xi|)^k|\partial^k_\xi e^{-t\xi^2}| \leqslant C(1+a)^k \sum_{0\leqslant l\leqslant\frac{k}{2}} a^{2k-3l}$. Therefore, we showed that $(1+|\xi|)^k|\partial^k_\xi e^{-t\xi^2}| \leqslant C_k$ uniformly on $t \in [0,a]$, hence $p_t \in S^0_{1,0}$ uniformly on $t \in [0,a]$. We also have for all $\delta, u > 0$, $t \leqslant \delta^{1+2u}$ implies that $\sup_\xi |(1+|\xi|)^{-u}(e^{-t\xi^2}-1)| \leqslant \delta$ which means that $\sup_\xi |(1+|\xi|)^{-u}(e^{-t\xi^2}-1)| \to 0$ when $t \to 0^+$ which implies the convergence $p_t \to 1$ in $S^{+0}_{1,0}$. By a result of Strichartz [79, Thm 1.3 p. 296],

$$p_t(\sqrt{\Delta}) = e^{-t\Delta} \underset{t\to 0^+}{\to} Id \text{ in } \Psi^{+0}_{1,0}(M). \tag{11.2}$$

□

11.4. **Proof of Lemma 4.19.** Without loss of generality we assume $0 \in \Omega$ and we try to prove the Lemma in some neighborhood of $0 \in \Omega$. For every fixed $V \in \Omega$, and for every complex $z \in \mathbb{C}$ small enough, $\partial^n_z F_1(zV) = \partial^n_z F_2(zV)$ by assumption, therefore the uniqueness of the Taylor series and its convergence for analytic functions of one variable yields the identity

$$F_1(zV) = P(z,V) + F_2(zV)$$

where both sides are holomorphic germs in $z$ near $0 \in \mathbb{C}$ and $P(z,V)$ is a polynomial in $z$ of degree $k-1$. The subtlety is that here we have an identity which holds true along every complex ray $\{zV, z \in \mathbb{C}\}$ in the open subset $\Omega \subset E$ and both sides are holomorphic functions of one variable $z \in \mathbb{C}$. We would like to deduce a similar identity without $z \in \mathbb{C}$ and were both sides are viewed as holomorphic functions on $C^\infty(End(E))$ in the sense of definition 2.2.

To finish the proof of the Lemma, we recall the definition of finitely holomorphic (also called Gâteaux–holomorphic) functions which is the weakest notion of holomorphicity in $\infty$–dimension [23, p. 54 def 2.2]:

**Definition 11.3** (Finitely holomorphic functions)**.** *Let $\Omega$ open in some Fréchet space $E$ over $\mathbb{C}$. A function $f : E \mapsto \mathbb{C}$ is said to be finitely holomorphic on $\Omega$ if for all $A \in \Omega$, every $B \in E$, $z \in \mathbb{C} \mapsto f(A + zB)$ is a* ***holomorphic germ*** *at $z = 0$.*

Beware that finitely holomorphic maps are not necessarily continuous since any $\mathbb{C}$-linear map $F : E \mapsto \mathbb{C}$ which is not even continuous is always finitely holomorphic. The notion of holomorphicity from definition 2.2 is the strongest possible and Fréchet holomorphic functions are automatically smooth hence $C^0$ unlike finitely holomorphic functions.

Our goal in this part is to recall the proof that finitely analytic maps near $A$ which are **locally bounded** are analytic near $A$.

**Definition 11.4** (Local boundedness)**.** *A map $f$ is locally bounded near $A$ if there is an open neighborhood $U \subset E$ of $A$ and $0 \leqslant M < +\infty$ such that $|f|_U| \leqslant M$.*

The proof is inspired from the thesis of Douady [24, Prop 2 p. 9] and also [23, p. 57–58].

**Proposition 11.5.** *Let $E$ be a Fréchet space and $F : E \mapsto \mathbb{C}$ finitely analytic on $\Omega \subset E$. If $F$ is locally bounded at $A$, in particular if on a ball $B(A, r) = \{B \text{ s.t. } \|B - A\| < r\}$ for a continuous norm $\|.\|$ on $E$, $\sup_{B(A,r)} |F| \leqslant M < +\infty$, then $F$ is Fréchet differentiable at $A$ at any order and can be identified with its Taylor series near $a$:*

$$F(A + H) = \sum_{n=0}^{\infty} P_n(H)$$

*where each $P_n$ is a continuous polynomial map homogeneous of degree $n$, the $P_n$ are uniquely determined by $P_n(h) = \frac{n!}{2i\pi} \int_\gamma F(A + \lambda h) \frac{d\lambda}{\lambda^{n=1}}$ and $\sum \|P_n\| \tilde{r}^n < +\infty$ for every $0 < \tilde{r} < r$.*

*In particular $F$ smooth in some neighborhood of $A$.*

*Proof.* This proposition is well–known when $B$ has finite dimension and the expansion $F(a + h) = \sum_n P_n(h)$ is given by the formula $P_n(h) = \frac{1}{2\pi} \int_0^{2\pi} F(a + e^{i\theta} h) e^{-in\theta} d\theta$ where $|h| \leqslant r$ and then we keep this formula in the infinite dimensional case. The integral is that of a continuous function (by finite analyticity) hence is well–defined. If $F$ is bounded by $M$ on a ball of radius $r > 0$ for the continuous norm $\|.\|$ then so is $P_n$. To show that $P_n$ is a homogeneous monomial, we follow Douady's approach by setting $\tilde{P}_n(h_1, \dots, h_n) = \frac{1}{n!} \Delta_{h_1} \dots \Delta_{h_n} P_n$ where $\Delta_h$ is the finite difference operator $\Delta_h P(x) = \frac{1}{2} (P(x + h) - P(x - h))$. In the finite dimensional case $\tilde{P}_n$ is multilinear and it is the same in the infinite dimensional case since it only depends on the restriction of $P_n$ to some finite dimensional subspace of $E$. Hence $P_n(h) = \tilde{P}_n(h, \dots, h)$ for some symmetric multilinear map $\tilde{P}_n$. From Cauchy's integral formula, we know that every $P_n$ is bounded by $M$ when $\|h\| \leqslant r$ which implies that $|\tilde{P}_n(h_1, \dots, h_n)| \leqslant \frac{n^n M}{n!} \|h_1\| \dots \|h_n\|$ hence $\tilde{P}_n$ is continuous. From this it results that the series $\sum_n P_n$ has normal convergence and the proposition is proved. $\square$

Since both $V \in \Omega \mapsto F_1(V)$ and $V \in \Omega \mapsto F_2(V)$ are locally bounded near $V = 0$ by holomorphicity of $F_1, F_2$, the above Proposition 11.5 applied to the holomorphic function $F_1 - F_2$ implies that we have the equality

$$F_1(V) = P(V) + F_2(V) \tag{11.3}$$

for $V$ close enough to 0 where $P$ is a **uniquely determined continuous polynomial function** of $V \in E$.

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