

A soft-photon theorem for the Maxwell-Lorentz system

Wojciech Dybalski

Zentrum Mathematik, Technische Universität München,

E-mail: dybalski@ma.tum.de

Duc Viet Hoang

Fakultät für Physik, Ludwig-Maximilians-Universität München

E-mail: V.hoang@physik.lmu.de

Abstract

For the coupled system of classical Maxwell-Lorentz equations we show that the quantities

$$\mathfrak{F}(\hat{x}, t) = \lim_{|x| \rightarrow \infty} |x|^2 F(x, t), \quad \mathcal{F}(\hat{k}, t) = \lim_{|k| \rightarrow 0} |k| \hat{F}(k, t),$$

where F is the Faraday tensor, \hat{F} its Fourier transform in space and $\hat{x} := \frac{x}{|x|}$, are independent of t . We combine this observation with the scattering theory for the Maxwell-Lorentz system due to Komech and Spohn, which gives the asymptotic decoupling of F into the scattered radiation $F_{\text{sc}, \pm}$ and the soliton field $F_{v_{\pm\infty}}$ depending on the asymptotic velocity $v_{\pm\infty}$ of the electron at large positive (+), resp. negative (-) times. This gives a *soft-photon theorem* of the form

$$\mathcal{F}_{\text{sc},+}(\hat{k}) - \mathcal{F}_{\text{sc},-}(\hat{k}) = -(\mathcal{F}_{v_{+\infty}}(\hat{k}) - \mathcal{F}_{v_{-\infty}}(\hat{k})),$$

and analogously for \mathfrak{F} , which links the low-frequency part of the scattered radiation to the change of the electron's velocity. Implications for the infrared problem in QED are discussed in the Conclusions.

1 Introduction

It is well known that a formal application of the Noether theorem to the global $U(1)$ symmetry of QED gives conservation of the electric charge. It is less well known that a similar reasoning applied to the local gauge symmetry ensures conservation of the spacelike asymptotic flux of the electric field¹

$$\phi(n) := \lim_{r \rightarrow \infty} r^2 n \cdot E(nr), \quad n \in S^2. \quad (1.1)$$

¹A heuristic argument can be found in <https://en.wikipedia.org/wiki/Infraparticle>.

The relevance of such asymptotic quantities for qualitative understanding of infrared problems has been known for long [Bu82]. Intriguing relations between asymptotic symmetries, soft-photon theorems and memory effects, recently pointed out by Strominger, led to a revival of interest in this subject (see [St17] for a review). Thus there is every reason to advance rigorous mathematical understanding of asymptotic constants of motion in classical and quantum electrodynamics.

In this paper we prove the existence of a large family of asymptotic constants of motion, including (1.1), for a classical system of coupled Maxwell-Lorentz equations, also known as the Abraham model (see Section 2). This system describes one (spatially extended) electron interacting via the Lorentz force with the electromagnetic field. The existence and uniqueness of solutions for this model and the long-time asymptotics was clarified by the works of Komech and Spohn [Sp, KS00] which provide the basis for the present investigation. These authors emphasized the role of the traveling wave (or soliton) solutions, which have the form

$$(E(x, t), B(x, t), q(t), v(t)) = (E_v(x - q - vt), B_v(x - q - vt), q + vt, v), \quad (1.2)$$

where E, B, q, v denote the electromagnetic fields, the position and velocity of the electron, respectively. E_v and B_v are concrete functions which can be obtained by minimizing the total energy of the system at fixed momentum. Of course, $E_{v=0}$ is simply the Coulomb field of the electron at rest. For arbitrary $|v| < 1$ we have (in Fourier space)

$$\hat{E}_v(k) = \frac{-ik + v(v \cdot ik)}{|k|^2 - (k \cdot v)^2} e\hat{\varphi}(k), \quad (1.3)$$

where $e\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ describes the charge distribution of the electron. For sufficiently small e and suitable initial data Komech and Spohn showed that

$$\lim_{t \rightarrow \pm\infty} \|E(\cdot, t) - E_{v(t)}(\cdot - q(t)) - E_{\text{sc}, \pm}(\cdot, t)\|_{L^2(\mathbb{R}^3)} = 0, \quad (1.4)$$

where the *scattered radiation fields* $E_{\text{sc}, \pm}$ satisfies the homogeneous Maxwell equations. The existence of the asymptotic velocities of the electron $v_{\pm\infty} = \lim_{t \rightarrow \pm\infty} v(t)$ is an important intermediate result from [Sp, KS00].

Let us now describe in more detail the findings of the present paper. Under slightly more restrictive assumptions than Komech and Spohn, we show that

$$\mathfrak{E}(\hat{x}, t) = \lim_{|x| \rightarrow \infty} |x|^2 E(x, t), \quad (1.5)$$

depends only on $\hat{x} := x/|x|$ and is a constant of motion. That is, if the limit exist for $t = 0$ then it exists for arbitrary $t \in \mathbb{R}$ and has the same value. Clearly, the spacelike asymptotic flux of the electric field (1.1) inherits the properties of \mathfrak{E} . For the discussion of infrared singularities it is convenient to have a counterpart of (1.5) in momentum space. We introduce

$$\mathcal{E}(\hat{k}, t) = \lim_{|k| \rightarrow 0} |k| \hat{E}(k, t), \quad (1.6)$$

which is a constant of motion in the same sense as \mathfrak{E} . We verify that it has a decomposition

$$\mathcal{E}(\hat{k}, \pm t) = \mathcal{E}_{\text{sc},\pm}(\hat{k}) + \mathcal{E}_{v_{\pm\infty}}(\hat{k}), \quad t \geq 0, \quad (1.7)$$

which reflects the asymptotic decoupling in (1.4) for the two time directions. Hence, due to the conservation of \mathcal{E} we obtain the following variant of the *soft-photon theorem*

$$\mathcal{E}_{\text{sc},+}(\hat{k}) - \mathcal{E}_{\text{sc},-}(\hat{k}) = -(\mathcal{E}_{v_{+\infty}}(\hat{k}) - \mathcal{E}_{v_{-\infty}}(\hat{k})) \quad (1.8)$$

which links the low-frequency part of the scattered radiation to the change of the electron's velocity. Thus it has a similar physical meaning as Weinberg's soft-photon theorem in QED (see e.g. [St17, formula (2.8.21)]) and there is currently substantial interest in the high energy physics community concerning related asymptotic conditions in classical electrodynamics (see e.g. [Pr18] and references therein). While similar relations are rigorously known in the external current situation, see e.g. [He17], we are not aware of such results for the Maxwell-Lorentz system. Since the r.h.s. of (1.8) can readily be computed using (1.3), we obtain non-trivial information about the timelike asymptotics of solutions of the coupled Maxwell-Lorentz system: Namely, at least one of the scattered fields $E_{\text{sc},\pm}$ (incoming or outgoing) must have a $1/|k|$ singularity for small $|k|$.

Although we consider only the Maxwell-Lorentz system with one electron in this paper, we expect that soft-photon theorems similar to (1.8) are true for a much larger class of systems. Natural future research directions include the case of many, possibly spinning electrons, as discussed in [Sp]. Actually, for any system of non-linear PDE, admitting long-time asymptotics in terms of soliton solutions, one can try to formulate and prove relations similar to (1.8).

This paper is organized as follows. In Section 2 we provide some background material about the existence of solutions and scattering theory of the Maxwell-Lorentz system. In Section 3 we prove the existence of the constant of motion (1.5) and in Section 4 of its counterpart in momentum space (1.6). Section 5 contains the proof of our main result, which is the soft-photon theorem (1.8). In the Conclusions we discuss briefly the infrared problem in QED from the perspective of our findings.

2 Preliminaries

We give an introduction to the Abraham model. All proofs can be found in [Sp, KS00, Ho18].

2.1 The equations of motion for the Maxwell-Lorentz system

We are interested in the dynamics of well-localized charges representing charged particles of finite extension. In the course of this discussion, let e be the *charge*

of such a particle and m its *mass*. The *position* $q : \mathbb{R} \rightarrow \mathbb{R}^3$ of the particle is a function of time $t \in \mathbb{R}$ and $\frac{d}{dt}q(t) \equiv \dot{q}(t) =: v(t)$ denotes its *velocity*. We always assume that $q \in \mathcal{C}^2(\mathbb{R})$ and $v \in \mathcal{C}^1(\mathbb{R})$, i.e., the second derivative of $t \mapsto q(t)$ and first derivative of $t \mapsto v(t)$ exist and are continuous for all $t \in \mathbb{R}$. The *electric field* E and the *magnetic field* B are represented as vector fields $E = (E_1, E_2, E_3)$ and $B = (B_1, B_2, B_3)$ with

$$E_i, B_i : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R} \quad \text{for } i = 1, 2, 3.$$

Moreover, we introduce $\varphi \in \mathcal{C}^\infty(\mathbb{R}^3)$ which is the *charge distribution* of the particle. That is, for all $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$

$$\begin{aligned} \rho(x, t) &= e\varphi(x - q(t)) && \text{is the charge density,} \\ j(x, t) &= e\varphi(x - q(t))v(t) && \text{is the current density.} \end{aligned}$$

We assume in the Abraham model that the charge distribution φ satisfies the following properties:

1. φ is radial, i.e. $\varphi(x) = \varphi(|x|)$ for all $x \in \mathbb{R}^3$,
2. φ is compactly supported, i.e. $\exists R_\varphi > 0$ such that for all $|x| \geq R_\varphi$: $\varphi \equiv 0$.
3. φ is normalized, i.e. $\int \varphi(x) d^3x = 1$.

As a result, we are able to couple the Maxwell equations with the Lorentz equation:

Definition 2.1. *We call*

$$\partial_t B(x, t) = -\nabla \times E(x, t), \tag{2.1}$$

$$\partial_t E(x, t) = \nabla \times B(x, t) - e\varphi(x - q(t))v(t), \tag{2.2}$$

$$\nabla \cdot E(x, t) = e\varphi(x - q(t)), \tag{2.3}$$

$$\nabla \cdot B(x, t) = 0, \tag{2.4}$$

$$\frac{d}{dt} \{m\gamma v(t)\} = e \{E_\varphi(q(t), t) + v(t) \times B_\varphi(q(t), t)\} \tag{2.5}$$

the equations of motion for the Abraham model. Here $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$, and $\gamma \equiv \gamma(t) := \frac{1}{\sqrt{1-v(t)^2}}$. In addition, the fields E and B are smeared with the function φ in the last equation, that is,

$$E_\varphi(x, t) \equiv (E(\cdot, t) * \varphi)(x), \quad B_\varphi(x, t) \equiv (B(\cdot, t) * \varphi)(x).$$

2.2 Existence of solutions of the Maxwell-Lorentz system

We will denote by $Y := (E, B, q, v)$ a state of the Abraham model, and define the set of states

$$\mathcal{L} := \{Y \in (L^2 \cap \mathcal{C}^2) \times (L^2 \cap \mathcal{C}^2) \times \mathbb{R}^3 \times \mathbb{V} \mid \|E\| + \|B\| + |q| + |\gamma v| < \infty\}, \tag{2.6}$$

where $\mathbb{V} := \{v \in \mathbb{R}^3 \mid |v| < 1\}$, $\|F\| := (\int d^3x |F(x)|^2)^{1/2}$ and \mathcal{C}^2 is the space of twice continuously differentiable functions. By imposing, in addition, the constraints on the set of states, we obtain the phase space of the model:

$$\mathcal{M} := \left\{ Y \in \mathcal{L} \mid \nabla \cdot E(x) = e\varphi(x - q), \nabla \cdot B(x) = 0 \quad \forall x \in \mathbb{R}^3 \right\}.$$

For future reference, we also introduce certain subsets of \mathcal{M} . For any $\sigma \in [0, 1]$ we say that $Y \in \mathcal{M}^\sigma$ if $Y \in \mathcal{M}$ and there exist $C, R > 0$ such that for all $|x| > R$

$$|E(x)| + |B(x)| + |x| (|\nabla E(x)| + |\nabla B(x)|) \leq \frac{C}{|x|^{1+\sigma}}.$$

We can rewrite the equation of motion for the Abraham model as a generalized differential equation

$$\frac{d}{dt} Y(t) = F(Y(t)), \quad Y(0) = Y^0$$

or equivalently via integration over the interval $[0, t]$, $t \in \mathbb{R}$:

$$Y(t) = Y^0 + \int_0^t F(Y(s)) ds.$$

Then we have the following statement:

Proposition 2.2. *[KS00, Sp] Let $Y^0 = (E^0, B^0, q^0, v^0) \in \mathcal{M}$. Then the integral equation associated with the equation of motion,*

$$Y(t) = Y^0 + \int_0^t F(Y(s)) ds,$$

has a unique solution $Y(t) = (E(\cdot, t), B(\cdot, t), q(t), v(t)) \in \mathcal{M}$ for all $t \in \mathbb{R}$, which is continuous in t and satisfies $Y(0) = Y^0$.

In [KS00] the phase space \mathcal{M} is defined without the restriction to \mathcal{C}^2 functions. However, it easily follows from formulas (2.16), (2.17) of [Sp], that for initial conditions from \mathcal{C}^2 also the solutions are \mathcal{C}^2 .

As the proof of Proposition 2.2 from [KS00, Sp] is restricted to the case $t \geq 0$, let us briefly discuss the general case. We recall that the proof of the proposition starts from the equation for the Lorentz force (2.5) and uses formulas (2.16), (2.17) of [Sp] to express the r.h.s. by integral formulas involving propagators of the wave equation. For example, for $t \geq 0$ one expresses the electric field as

$$E(t) = \partial_t G_{\text{ret},t} * E^0 + \nabla \times (G_{\text{ret},t} * B^0) \tag{2.7}$$

$$- \int_0^t ds (\nabla G_{\text{ret},t-s} * \rho(s) + \partial_t G_{\text{ret},t-s} * j(s)), \tag{2.8}$$

where we omitted the x variable and $G_{\text{ret},t}(x) := \theta(t) \frac{1}{4\pi t} \delta(|x| - t)$ is the retarded propagator of the wave equation. For $t \leq 0$ this equation should be modified to

$$E(t) = \partial_t(-G_{\text{adv},t}) * E^0 + \nabla \times (-G_{\text{adv},t} * B^0) \quad (2.9)$$

$$- \int_0^t ds (\nabla G_{\text{adv},t-s} * \rho(s) + \partial_t G_{\text{adv},t-s} * j(s)), \quad (2.10)$$

where $G_{\text{adv},t}(x) := G_{\text{ret},-t}(x) = -\theta(-t) \frac{\delta(|x|+t)}{4\pi t}$ is the advanced propagator and the minus sign in (2.9), compared to (2.7), is needed to match the two formulas at $t = 0$. (Of course, the parts (2.8) and (2.10) cannot have such a relative minus sign, as they have to satisfy the same inhomogeneous wave equation). Thus a formula valid for all $t \in \mathbb{R}$ has the form

$$E(t) = \partial_t G_t * E^0 + \nabla \times (G_t * B^0) \quad (2.11)$$

$$- \int_0^t ds (\nabla G_{\text{ret/adv},t-s} * \rho(s) + \partial_t G_{\text{ret/adv},t-s} * j(s)), \quad (2.12)$$

where $G_t := G_{\text{ret},t} - G_{\text{adv},t}$ is the causal propagator and the choice of the retarded or advanced propagator in (2.12) is correlated with the sign of t . The estimates involved in the later application of the Banach fixed point theorem in the proof of Proposition 2.2 are insensitive to the changes of the propagator discussed above. Thus one obtains a unique trajectory $\mathbb{R} \ni t \mapsto q(t)$ and, via (2.11)-(2.12), the electric field. The magnetic field is treated analogously. Existence of the second derivative of $t \mapsto q(t)$ and the first derivative of $t \mapsto v(t)$ at $t = 0$ follows from the Lorentz force equation and the fact that the electromagnetic fields tend to E^0, B^0 as $t \rightarrow 0$ both along the negative and positive values of t . See also [Ha18] for the problem of existence of solutions of the Maxwell-Lorentz system for point charges for positive and negative times.

2.3 Soliton solutions

Proposition 2.2 gives the existence and uniqueness, but no concrete construction of solutions. To proceed, we consider a specific class of solutions, namely the *charge solitons* which represent charged particles with constant velocities $v \in \mathbb{V}$. First, we define the following electromagnetic fields:

Definition 2.3. We set for all $x \in \mathbb{R}^3 \setminus \{0\}$

$$\phi_v(x) := \frac{e}{4\pi \sqrt{(x/\gamma)^2 + (v \cdot x)^2}}, \quad \phi_{v\varphi}(x) := (2\pi)^{-3/2} \phi_v * \varphi(x)$$

and define the electric field E_v of a soliton with velocity v and the magnetic field B_v of a soliton with velocity $v \in \mathbb{V}$ as

$$E_v(x) = -\nabla \phi_{v\varphi}(x) + v (v \cdot \nabla \phi_{v\varphi}(x)), \quad B_v(x) = -v \times \nabla \phi_{v\varphi}(x). \quad (2.13)$$

The corresponding momentum space expressions are

$$\hat{E}_v(k) = -ik\hat{\phi}_{v\varphi} + v(v \cdot ik\hat{\phi}_{v\varphi}), \quad \hat{B}_v(k) = -v \times (ik\hat{\phi}_{v\varphi}), \quad (2.14)$$

where

$$\hat{\phi}_v \equiv \hat{\phi}_v(k) = \frac{e}{k^2 - (k \cdot v)^2}, \quad \hat{\phi}_{v\varphi} \equiv \hat{\phi}_{v\varphi}(k) = \hat{\phi}_v(k)\hat{\varphi}(k). \quad (2.15)$$

(Note that $\phi_v \notin L^1(\mathbb{R}^3, \mathbb{R})$, so the Fourier transformations² need to be understood in the distributional sense). Now the soliton solution is constructed as follows:

Proposition 2.4. *[KS00, Sp] For any $v \in \mathbb{V}$, the following family of states*

$$Y_v(t) = (E_v(\cdot - q - vt), B_v(\cdot - q - vt), q + vt, v) \in \mathcal{M}$$

is a solution of the Abraham model.

2.4 Long-time asymptotics

It turns out that a large class of solutions converges to soliton solutions for large times via emission of radiation. It is therefore convenient to define

$$Z(x, t) = \begin{pmatrix} Z_1(x, t) \\ Z_2(x, t) \end{pmatrix} = \begin{pmatrix} E(x, t) - E_{v(t)}(x - q(t)) \\ B(x, t) - B_{v(t)}(x - q(t)) \end{pmatrix}, \quad (2.16)$$

where $(E_{v(t)}(x - q(t)), B_{v(t)}(x - q(t)), q(t), v(t))$ is the soliton approximation at time $t \in \mathbb{R}$. We remark for future reference that the system of equations of the Abraham model can then be rewritten in the following form [KS00, Sp]

$$Z(x, t) = \mathbf{U}(t)Z(x, 0) - \int_0^t ds \mathbf{U}(t - s)\mathbf{g}(x, s), \quad (2.17)$$

where

$$\mathbf{g}(x, t) = \begin{pmatrix} (\dot{v}(t) \cdot \nabla_v)E_{v(t)}(x - q(t)) \\ (\dot{v}(t) \cdot \nabla_v)B_{v(t)}(x - q(t)) \end{pmatrix}.$$

The operator \mathbf{U} is unitary on $L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ and is given by the solutions of the homogeneous Maxwell's equations in position space:

$$\mathbf{U}(t)\mathbf{F}(x, 0) \equiv \begin{pmatrix} \{\partial_t G_t * F_1(\cdot, 0)\}(x) + \nabla \times \{G_t * F_2(\cdot, 0)\}(x) \\ \{\partial_t G_t * F_2(\cdot, 0)\}(x) - \nabla \times \{G_t * F_1(\cdot, 0)\}(x) \end{pmatrix},$$

where F_1, F_2 are the components of \mathbf{F} and $G_t(x) = \theta(t)\frac{1}{4\pi t}\delta(|x| - t) + \theta(-t)\frac{\delta(|x| + t)}{4\pi t}$ is the causal propagator. We note for future reference that in momentum space

$$\hat{G}_t(k) = (2\pi)^{-3/2} \frac{\sin(|k|t)}{|k|} \quad \text{for } t \in \mathbb{R}. \quad (2.18)$$

²We use the conventions for the Fourier transform from [RS2].

The advantage of (2.17) is that the only implicit input on the r.h.s. is $t \mapsto (q(t), v(t))$.

For large positive (+) or negative (-) times, the differences on the r.h.s. of (2.16) should give the scattered radiation field. Thus we write $Z_{\text{sc},\pm}(x, t) = (E_{\text{sc},\pm}(x, t), B_{\text{sc},\pm}(x, t))$, where

$$Z_{\text{sc},\pm}(x, 0) := Z(x, 0) - \int_0^{\pm\infty} ds \mathbf{U}(-s)\mathbf{g}(x, s), \quad (2.19)$$

$$Z_{\text{sc},\pm}(x, t) := \mathbf{U}(t)Z_{\text{sc},\pm}(x, 0). \quad (2.20)$$

By definition, $(x, t) \mapsto (E_{\text{sc},\pm}(x, t), B_{\text{sc},\pm}(x, t))$ satisfy the homogeneous Maxwell equations. The long-time asymptotics of the Abraham model is now described by the following theorem. We remark that the proof given in [KS00, Sp] for $t \geq 0$ generalizes to $t \leq 0$, since the relevant estimates are not sensitive to the substitution $G_{\text{ret}} \rightarrow -G_{\text{adv}}$, cf. also formula (2.18).

Theorem 2.5. [KS00, Sp] *For $|e| \leq \bar{e}$, where \bar{e} is sufficiently small, and $Y(0) \in \mathcal{M}^\sigma$ for $\sigma \in (0, 1]$ the following statements hold true:*

1. *The acceleration \dot{v} satisfies $|\dot{v}(t)| \leq C(1 + |t|)^{-1-\sigma}$ for some C independent of t . Hence, $\lim_{t \rightarrow \pm\infty} v(t) = v_{\pm\infty} \in \mathbb{V}$ exists.*
2. *For $(E_{\text{sc},\pm}, B_{\text{sc},\pm})$ defined in (2.19), (2.20), we have*

$$\lim_{t \rightarrow \pm\infty} \left\{ \|E(\cdot, t) - E_{v(t)}(\cdot - q(t)) - E_{\text{sc},\pm}(\cdot, t)\| \right. \quad (2.21)$$

$$\left. + \|B(\cdot, t) - B_{v(t)}(\cdot - q(t)) - B_{\text{sc},\pm}(\cdot, t)\| \right\} = 0. \quad (2.22)$$

(The \pm -signs above are correlated).

The assumptions of Theorem 2.5 are our standing assumptions in the remaining part of this paper.

3 A conservation law in position space

We denote by F the Faraday tensor, whose components are the electromagnetic fields E_i, B_j , $i, j = 1, 2, 3$. We set $\mathfrak{F}(x, t) := \lim_{|x| \rightarrow \infty} |x|^2 F(x, t)$ (if the limit exists) and denote by $\mathfrak{E}, \mathfrak{B}$ its electric and magnetic components.

Theorem 3.1. *Suppose that $\mathfrak{F}(x, 0) := \lim_{|x| \rightarrow \infty} |x|^2 F(x, 0)$ exists and depends only on $\hat{x} = \frac{x}{|x|}$. Then also $\mathfrak{F}(\hat{x}, t) := \lim_{|x| \rightarrow \infty} |x|^2 F(x, t)$ exists for any $t \in \mathbb{R}$ and depends only on \hat{x} in the first variable. Moreover, $\mathfrak{F}(\hat{x}, t) = \mathfrak{F}(\hat{x}, 0)$.*

Remark 3.2. *In $\mathcal{M}^{\sigma=1}$ there are many examples of initial data with slow, Coulomb-type decay for which $\mathfrak{F}(\hat{x}, 0)$ exists.*

Proof. We treat only the electric field for $t > 0$, as the remaining cases are analogous. By (2.17), we have

$$E(x, t) = E_1(x, t) + E_2(x, t) + E_3(x, t) + E_4(x, t) + E_{v(t)}(x - q(t)) \quad (3.1)$$

with

$$\begin{aligned} E_1(x, t) &:= \partial_t G_t * [E(\cdot, 0) - E_{v(0)}(\cdot - q(0))] (x), \\ E_2(x, t) &:= \nabla \times \left\{ G_t * [B(x, 0) - B_{v(0)}(x - q(0))] \right\}, \\ E_3(x, t) &:= - \int_0^t ds \left[\partial_\tau G_\tau \Big|_{\tau=t-s} * (\dot{v}(s) \cdot \nabla_v) E_{v(s)}(\cdot - q(s))(x) \right], \\ E_4(x, t) &:= - \int_0^t ds \left[\nabla \times \left\{ G_\tau \Big|_{\tau=t-s} * (\dot{v}(s) \cdot \nabla_v) B_{v(s)}(\cdot - q(s))(x) \right\} \right]. \end{aligned}$$

Let us consider the contribution of E_1 . We obtain the chain of equalities below, which gives the existence of $\lim_{|x| \rightarrow \infty} |x|^2 E_1(x, t)$, using the existence of an analogous limit for the initial data and a soliton solution. We will use in the first step the limit $|x| \rightarrow \infty$ eliminates the $q(0)$ dependence. We will also repetitively apply the dominated convergence theorem to exchange the limit with integrals. Its assumptions are verified using concrete formulas for the soliton fields and the fact that the initial data are in \mathcal{M}^σ , $\sigma \in (0, 1]$.

$$\begin{aligned} & \lim_{|x| \rightarrow \infty} |x|^2 E_1(x, t) \\ &= - \lim_{|x| \rightarrow \infty} |x|^2 \int d^3 y \frac{\delta'(|y| - t)}{4\pi|y|} [E(x - y, 0) - E_{v_0}(x - y)] \\ &= \lim_{|x| \rightarrow \infty} |x|^2 \int d\Omega(\hat{y}) \int d|y| \frac{\delta(|y| - t)}{4\pi} \frac{d}{d|y|} [|y| (E(x - y, 0) - E_{v_0}(x - y))] \\ &= \lim_{|x| \rightarrow \infty} |x|^2 \frac{1}{4\pi} \int d\Omega(\hat{y}) [(E(x - \hat{y}t, 0) - E_{v_0}(x - \hat{y}t)) + t (E'(x - \hat{y}t, 0) - E'_{v_0}(x - \hat{y}t))] \\ &\stackrel{(*)}{=} \frac{1}{4\pi} \int d\Omega(\hat{y}) \left[\lim_{|x| \rightarrow \infty} |x|^2 E(x - \hat{y}t, 0) - \lim_{|x| \rightarrow \infty} |x|^2 E_{v_0}(x - \hat{y}t) \right] \\ &= \frac{1}{4\pi} \int d\Omega(\hat{y}) \left[\lim_{|x| \rightarrow \infty} |x|^2 E \left(|x| \left(x - \frac{\hat{y}t}{|x|} \right), 0 \right) - \lim_{|x| \rightarrow \infty} |x|^2 E_{v_0} \left(|x| \left(x - \frac{\hat{y}t}{|x|} \right) \right) \right] \\ &= \lim_{|x| \rightarrow \infty} |x|^2 E(x, 0) - \lim_{|x| \rightarrow \infty} |x|^2 E_{v_0}(x). \end{aligned}$$

The prime above denotes the derivative w.r.t. $|y|$ and in (*) we used that $E'(x, 0)$, $E'_{v_0}(x) \sim \frac{1}{|x|^{2+\sigma}}$ for $\sigma > 0$ as the initial data and the soliton solutions belong to \mathcal{M}^σ for $\sigma \in (0, 1]$.

As for E_2 , we find

$$\begin{aligned}
\lim_{|x| \rightarrow \infty} |x|^2 E_2(x, t) &= \lim_{|x| \rightarrow \infty} |x|^2 \int d^3 y \frac{\delta(|y| - t)}{4\pi|y|} \nabla \times [B(x - y, 0) - B_{v_0}(x - y)] \\
&= \lim_{|x| \rightarrow \infty} |x|^2 \frac{1}{4\pi} \int d\Omega(\hat{y}) t \nabla \times [B(x - \hat{y}t, 0) - B_{v_0}(x - \hat{y}t)] \\
&= \frac{1}{4\pi} \int d\Omega(\hat{y}) t \lim_{|x| \rightarrow \infty} |x|^2 \nabla \times [B(x - \hat{y}t, 0) - B_{v_0}(x - \hat{y}t)] \\
&= 0,
\end{aligned}$$

where we again made use of the fact that the initial data and the soliton solutions belong to \mathcal{M}^σ .

Let us move on to the contribution of E_3 . First, we write

$$\lim_{|x| \rightarrow \infty} |x|^2 E_3(x, t) = - \lim_{|x| \rightarrow \infty} |x|^2 \int_0^t ds \int d^3 y \frac{(-1)}{4\pi|y|} \delta'(|y| - (t - s)) (\dot{v}(s) \cdot \nabla_v) E_{v(s)}(x - y),$$

where we used that the limit (if it exists) eliminates the $q(s)$ dependence. Next, we note the equality

$$\begin{aligned}
&\int d^3 y \frac{(-1)}{4\pi|y|} \delta'(|y| - (t - s)) (\dot{v}(s) \cdot \nabla_v) E_{v(s)}(x - y) \\
&= \frac{1}{4\pi} \int d\Omega(\hat{y}) [(t - s) (\dot{v}(s) \cdot \nabla_v) E'_{v(s)}(x - \hat{y}(t - s)) + (\dot{v}(s) \cdot \nabla_v) E_{v(s)}(x - \hat{y}(t - s))].
\end{aligned}$$

Hence, making use of the rapid decay of $E'_{v(s)}$, we find

$$\begin{aligned}
\lim_{|x| \rightarrow \infty} |x|^2 E_3(x, t) &= \frac{-1}{4\pi} \int d\Omega(\hat{y}) \lim_{|x| \rightarrow \infty} |x|^2 \int_0^t ds (\dot{v}(s) \cdot \nabla_v) E_{v(s)}(x - \hat{y}(t - s)) \\
&= \frac{-1}{4\pi} \int d\Omega(\hat{y}) \lim_{|x| \rightarrow \infty} |x|^2 \int_0^t ds (\dot{v}(s) \cdot \nabla_v) E_{v(s)}(x) \\
&= \frac{-1}{4\pi} \int d\Omega(\hat{y}) \left[\lim_{|x| \rightarrow \infty} |x|^2 E_{v(t)}(x) - \lim_{|x| \rightarrow \infty} |x|^2 E_{v_0}(x) \right] \\
&= - \lim_{|x| \rightarrow \infty} |x|^2 E_{v(t)}(x) + \lim_{|x| \rightarrow \infty} |x|^2 E_{v_0}(x),
\end{aligned}$$

where we used the Fubini theorem in the first step, in the second step we used the presence of the limit to eliminate the shift by $\hat{y}(t - s)$ and in the third step we noted that the integral w.r.t. s can be computed. By reading the above computation backwards, we obtain the existence of $\lim_{|x| \rightarrow \infty} |x|^2 E_3(x, t)$.

As for E_4 , we find

$$\begin{aligned}
&\lim_{|x| \rightarrow \infty} |x|^2 E_4(x, t) \\
&= - \lim_{|x| \rightarrow \infty} |x|^2 \frac{1}{4\pi} \int_0^t ds \int d\Omega(\hat{y}) (t - s) (\dot{v}(s) \cdot \nabla_v) \nabla \times B_{v(s)}(x - \hat{y}(t - s) - q(s)) = 0
\end{aligned}$$

since $\nabla \times B_{v(t)}(x) \sim \frac{1}{|x|^{2+\sigma}}$ for $\sigma > 0$.

By substituting all the contributions above to (3.1), we obtain

$$\lim_{|x| \rightarrow \infty} |x|^2 E(x, t) = \lim_{|x| \rightarrow \infty} |x|^2 E(x, 0).$$

This completes the proof. \square

4 A conservation law in momentum space

We recall that F is the Faraday tensor and we set $\mathcal{F}(k, t) := \lim_{|k| \rightarrow 0} |k| \hat{F}(k, t)$ (if the limit exists), where \hat{F} is the Fourier transform of F in space. We denote by \mathcal{E} , \mathcal{B} the electric and magnetic components of \mathcal{F} .

Theorem 4.1. *Suppose that $\mathcal{F}(k, 0) := \lim_{|k| \rightarrow 0} |k| \hat{F}(k, 0)$ exists and depends only on $\hat{k} = \frac{k}{|k|}$. Then also $\mathcal{F}(k, t) := \lim_{|k| \rightarrow \infty} |k| \hat{F}(k, t)$ exists for any $t \in \mathbb{R}$ and depends only on \hat{k} in the first variable. Moreover, $\mathcal{F}(\hat{k}, t) = \mathcal{F}(\hat{k}, 0)$.*

Proof. Again, we consider only the electric field and the case $t > 0$. We recall formula (2.17)

$$Z(x, t) = U(t)Z(x, 0) - \int_0^t ds U(t-s)g(x, s). \quad (4.1)$$

The electric part has the form considered already in the proof of the previous theorem:

$$\begin{aligned} E(x, t) = & \partial_t G_t * [E(\cdot, 0) - E_{v(0)}(\cdot - q(0))] (x) \\ & + \nabla \times \{G_t * [B(x, 0) - B_{v(0)}(x - q(0))]\} \\ & - \int_0^t ds \left[\partial_\tau G_\tau \Big|_{\tau=t-s} * (\dot{v}(s) \cdot \nabla_v) E_{v(s)}(\cdot - q(s))(x) \right. \\ & \quad \left. - \nabla \times \left\{ G_\tau \Big|_{\tau=t-s} * (\dot{v}(s) \cdot \nabla_v) B_{v(s)}(\cdot - q(s))(x) \right\} \right] \\ & + E_{v(t)}(x - q(t)). \end{aligned}$$

Using (2.18), we obtain in momentum space

$$\begin{aligned} \hat{E}(k, t) = & \cos(|k|t) \left[\hat{E}(k, 0) - \hat{E}_{v(0)}(k) e^{-ikq(0)} \right] \\ & + i\hat{k} \times \left\{ \sin(|k|t) \left[\hat{B}(k, 0) - \hat{B}_{v(0)}(k) e^{-ikq(0)} \right] \right\} \\ & - \int_0^t ds \left[\cos(|k|(t-s)) (\dot{v}(s) \cdot \nabla_v) \hat{E}_{v(s)}(k) e^{-ikq(s)} \right. \\ & \quad \left. - i\hat{k} \times \left\{ \sin(|k|(t-s)) (\dot{v}(s) \cdot \nabla_v) \hat{B}_{v(s)}(k) e^{-ikq(s)} \right\} \right] \\ & + \hat{E}_{v(t)}(k) e^{-ikq(t)}. \end{aligned}$$

Hence it holds

$$\begin{aligned}
\mathcal{E}(\hat{k}, t) &= \mathcal{E}(\hat{k}, 0) - \mathcal{E}_{v_0}(\hat{k}) + \mathcal{E}_{v(t)}(\hat{k}) \\
&\quad - \lim_{|k| \rightarrow 0} \int_0^t ds \left[\cos(|k|(t-s)) (\dot{v}(s) \cdot \nabla_v) |k| \hat{E}_{v(s)}(k) e^{-ikq(s)} \right. \\
&\quad \left. - i\hat{k} \times \left\{ \sin(|k|(t-s)) (\dot{v}(s) \cdot \nabla_v) |k| \hat{B}_{v(s)}(k) e^{-ikq(s)} \right\} \right] \quad (4.2) \\
&= \mathcal{E}(\hat{k}, 0) - \mathcal{E}_{v_0}(\hat{k}) - \int_0^t ds \left[(\dot{v}(s) \cdot \nabla_v) \mathcal{E}_{v(s)}(\hat{k}) \right] + \mathcal{E}_{v(t)}(\hat{k}) \\
&= \mathcal{E}(\hat{k}, 0),
\end{aligned}$$

where we used in the second step $\lim_{|k| \rightarrow 0} e^{-ikq(s)} = 1$, $\lim_{|k| \rightarrow 0} \cos(|k|(t-s)) = 1$, $\lim_{|k| \rightarrow 0} \sin(|k|(t-s)) = 0$ and in the last step we noted that the integral w.r.t. s can be evaluated. This concludes the proof. \square

5 Soft-photon theorem

Now we are ready to state and prove our main result. We consider a solution $\mathbb{R} \ni t \mapsto Y(t)$ of the Maxwell-Lorentz system satisfying the assumptions of Theorem 2.5. We denote by $F_{\text{sc}, \pm}$ the scattered electromagnetic fields and by $F_{v_{\pm\infty}}$ the soliton solutions corresponding to the asymptotic velocities.

Theorem 5.1. *Under the assumptions of Theorem 4.1 the limits*

$$\mathcal{F}_{\text{sc}, \pm}(k, t) = \lim_{|k| \rightarrow 0} |k| F_{\text{sc}}(k, t), \quad \mathcal{F}_{v_{\pm\infty}}(k, t) = \lim_{|k| \rightarrow 0} |k| F_{v_{\pm\infty}}(k, t) \quad (5.1)$$

exist, depend only on \hat{k} in the first variable and are independent of t . Moreover,

$$\mathcal{F}_{\text{sc}, +}(\hat{k}) + \mathcal{F}_{v_{+\infty}}(\hat{k}) = \mathcal{F}_{\text{sc}, -}(\hat{k}) + \mathcal{F}_{v_{-\infty}}(\hat{k}). \quad (5.2)$$

Remark 5.2. *An analogous statement holds for \mathfrak{F} .*

Proof. We provide the details only for the case of the electric field. For $t > 0$, the difference $Z(x, t) - Z_{\text{sc}}(x, t)$ yields by (2.17), (2.20),

$$\begin{aligned}
E(x, t) - E_{\text{sc}, +}(x, t) &= \int_t^\infty ds \left[\partial_\tau G_\tau \Big|_{\tau=t-s} * (\dot{v}(s) \cdot \nabla_v) E_{v(s)}(\cdot - q(s))(x) \right. \\
&\quad \left. \nabla \times \left\{ G_\tau \Big|_{\tau=t-s} * (\dot{v}(s) \cdot \nabla_v) B_{v(s)}(\cdot - q(s))(x) \right\} \right] \\
&\quad + E_{v(t)}(x - q(t)). \quad (5.3)
\end{aligned}$$

Thus, we find by analogous arguments as in the proof of Theorem 4.1, and making use of the fact that $t \mapsto \dot{v}(t)$ is integrable (see Theorem 2.5), that

$$\mathcal{E}(\hat{k}) - \mathcal{E}_{v(t)}(\hat{k}) - \mathcal{E}_{\text{sc}, +}(\hat{k}) = \mathcal{E}_{v_{+\infty}}(\hat{k}) - \mathcal{E}_{v(t)}(\hat{k}), \quad (5.4)$$

where all the taken limits exist and depend only on the indicated variables. Consequently,

$$\mathcal{E}(\hat{k}) = \mathcal{E}_{\text{sc},+}(\hat{k}) + \mathcal{E}_{v_{+\infty}}(\hat{k}) = \mathcal{E}_{\text{sc},-}(\hat{k}) + \mathcal{E}_{v_{-\infty}}(\hat{k}), \quad (5.5)$$

where in the last step we repeated the above reasoning for $t < 0$ and made use of the fact that $\mathcal{E}(\hat{k})$ is a conserved quantity. \square

6 Conclusions

In this paper we derived the following soft-photon theorem for the Maxwell-Lorentz system

$$\mathcal{E}_{\text{sc},+}(\hat{k}) - \mathcal{E}_{\text{sc},-}(\hat{k}) = -(\mathcal{E}_{v_{+\infty}}(\hat{k}) - \mathcal{E}_{v_{-\infty}}(\hat{k})), \quad (6.1)$$

and analogously for the magnetic field. In order to elucidate its physical meaning, let us consider a scattering process in which the electron, which is initially at rest ($v_{-\infty} = 0$), collides with infrared regular radiation ($\mathcal{E}_{\text{sc},-}(\hat{k}) = 0$). Making use of relation (1.3) and considering that the l.h.s. of (6.1) is transverse, we obtain

$$\mathcal{E}_{\text{sc},+}(\hat{k}) = -P_{\text{tr}}(\hat{k})\mathcal{E}_{v_{+\infty}}(\hat{k}), \quad (6.2)$$

where $P_{\text{tr}}(\hat{k}) := 1 - |\hat{k}\rangle\langle\hat{k}|$ is the transverse projection. This formula nicely brings together two seemingly distinct aspects of the infrared problem: the slow decay of the Coulomb field on the r.h.s. and the infrared singularity of the scattered radiation on the l.h.s. It is clear from our discussion, that the presence of the asymptotic constants of motion (1.6) is behind this equality. Making use of relation (1.3), we can rewrite (6.2) as follows

$$\lim_{|k|\rightarrow 0} |k|\hat{E}_{\text{sc},+}(k, t) = -\frac{ie}{(2\pi)^{3/2}} \left(\frac{(P_{\text{tr}}(\hat{k})v_{\infty})(\hat{k} \cdot v_{\infty})}{1 - (\hat{k} \cdot v_{\infty})^2} \right) \quad (6.3)$$

for any $t \geq 0$. This brings to light the infrared singularity $\hat{E}_{\text{sc}}(k, t) \sim 1/|k|$ of the scattered radiation. By an analogous discussion for the magnetic field we get

$$\lim_{|k|\rightarrow 0} |k|\hat{B}_{\text{sc},+}(k, t) = \frac{e}{(2\pi)^{3/2}} \frac{v_{\infty} \times i\hat{k}}{1 - (\hat{k} \cdot v_{\infty})^2}, \quad (6.4)$$

thus it also exhibits a $1/|k|$ behavior for small momenta.

It is customary to say that such radiation fields ‘escape from the Fock space’ of the quantised variant of our theory. Although such infrared jargon may be clear for experts, let us take the opportunity here to explain it from the perspective of the present investigation. Consider the free, second-quantised electromagnetic fields (in Fourier space), which are operator valued distributions on Fock space

$$\hat{E}(k, t) := \sum_{\lambda=\pm} \sqrt{\frac{|k|}{2}} (i\varepsilon_{\lambda}(k)e^{-i|k|t}a_{\lambda}(k) - i\varepsilon_{\lambda}(-k)e^{i|k|t}a_{\lambda}^*(-k)), \quad (6.5)$$

$$\hat{B}(k, t) := \sum_{\lambda=\pm} \sqrt{\frac{1}{2|k|}} ik \times (\varepsilon_{\lambda}(k)e^{-i|k|t}a_{\lambda}(k) + \varepsilon_{\lambda}(-k)e^{i|k|t}a_{\lambda}^*(-k)). \quad (6.6)$$

Here $a_\lambda^{(*)}(k)$ are the creation and annihilation operators of photons with momentum k and polarisation λ , $\varepsilon_\lambda(k)$ are the polarisation vectors and we refer e.g. to [Sp] for more details on these formulas. We interpret these fields as the (outgoing) asymptotic fields of the theory. To study the classical limit of this quantum theory, we consider expectation values of (6.5), (6.6) on coherent states of the form

$$|w\rangle = \exp\left(\sum_{\lambda=\pm} \int d^3k (w(k) \cdot \varepsilon_\lambda(k) a_\lambda^*(k) - \text{h.c.})\right) |0\rangle, \quad (6.7)$$

where $|0\rangle$ is the Fock space vacuum. (For an extensive discussion of the role of coherent states in the study of infrared problems we refer to [FMS79]). Specifically, we are looking for such \mathbb{C}^3 -valued functions w that this expectation value reproduces the classical low-energy behaviour (6.3), (6.4)

$$\lim_{|k|\rightarrow 0} |k| \langle w | \hat{\mathbf{E}}(k, t) | w \rangle = \lim_{|k|\rightarrow 0} |k| \hat{E}_{\text{sc},+}(k, t), \quad (6.8)$$

$$\lim_{|k|\rightarrow 0} |k| \langle w | \hat{\mathbf{B}}(k, t) | w \rangle = \lim_{|k|\rightarrow 0} |k| \hat{B}_{\text{sc},+}(k, t), \quad (6.9)$$

for $t \geq 0$. The l.h.s. of (6.8), (6.9) are readily computed and the equality holds for w of the familiar form, see e.g. [Fr73, Fr74.1, CFP09, CFP07, FP10, KM13],

$$w(k) = -\frac{e\hat{\varphi}(k)}{\sqrt{2}|k|^{3/2}} \frac{v_\infty}{1 - \hat{k} \cdot v_\infty}. \quad (6.10)$$

As the $|k|^{-3/2}$ singularity in (6.10) is not square integrable, indeed $|w\rangle$ given by (6.7) is not a vector in Fock space, but only makes sense as a state $\langle w | \cdot | w \rangle$ on the algebra of the electromagnetic fields. Via the GNS construction it gives a representation of this algebra which is disjoint from the Fock representation. In this sense the scattered field escapes from the Fock space.

Since $E_{\text{sc},\pm}$ appear in (1.4) as long-time asymptotics of the solutions of the Maxwell-Lorentz system, one can ask if $|w\rangle$ can be obtained by scattering theory. This turns out to be the case, as we have the equality (up to a phase), which can be traced back to Bloch and Nordsieck [BN37, BN37.1]

$$|w\rangle = W^+ |0\rangle, \quad W^+ = \text{Texp}\left(i \int_0^\infty dt \int d^3x \mathbf{A}(x, t) \cdot j_{v_\infty}(x, t)\right). \quad (6.11)$$

Here \mathbf{A} is the electromagnetic potential in the Coulomb gauge and $j_{v_\infty}(x, t) := ev_\infty \varphi(x - v_\infty t)$ is the external current describing the electron moving with velocity v_∞ . We recognize that W^+ is (up to a phase) the outgoing Dyson wave operator of the second-quantised electromagnetic field interacting with such a current. Here the infrared singularity is hidden in a possible divergence of the time-integral and the equality in (6.11) is meant in the sense of states on the algebra of the electromagnetic fields. Formula (6.11) is at the basis of the Faddeev-Kulish approach [KF70, Fr73, Dy17], as it ensures that the Dollard modifier maps the Fock space vacuum to the right representation of this algebra. A variant of this formula

was also used in [CD19] to show that this representation cannot be distinguished from the vacuum upon restriction to the interior of a future or backward light-cone. This latter observation supports the Buchholz-Roberts approach to infrared problems [BR14].

Acknowledgements: W.D. would like to thank Herbert Spohn for useful comments. Thanks are also due to Annika Steibel for helpful discussions on the existence and uniqueness of solutions of the Maxwell-Lorentz system. This work was partially supported by the Deutsche Forschungsgemeinschaft (DFG) within the Emmy Noether grant DY107/2-1.

References

- [BN37] F. Bloch and A. Nordsieck. *Note on the radiation field of the electron*. Phys. Rev. **52**, (1937) 54–59.
- [BN37.1] F. Bloch and A. Nordsieck. *The low frequency radiation of a scattered electron*. Phys. Rev. **52**, (1937) 59–62.
- [Bu82] D. Buchholz. *The physical state space of quantum electrodynamics*. Commun. Math. Phys. **85**, (1982) 49–71.
- [BR14] D. Buchholz and J.E. Roberts. *New light on infrared problems: sectors, statistics, symmetries and spectrum*. Commun. Math. Phys. **330**, (2014) 935–972.
- [CD19] D. Cadamuro and W. Dybalski. *Curing velocity superselection in non-relativistic QED by restriction to a lightcone*. arXiv:1902.09478.
- [CFP09] T. Chen, J. Fröhlich and A. Pizzo. *Infraparticle scattering states in non-relativistic QED: II. Mass shell properties*. J. Math. Phys. **50**, (2009) 012103.
- [CFP07] T. Chen, J. Fröhlich and A. Pizzo. *Infraparticle scattering states in non-relativistic QED: I. The Bloch-Nordsieck paradigm*. Commun. Math. Phys. **294**, (2010) 761–825.
- [Dy17] W. Dybalski. *From Faddeev-Kulish to LSZ. Towards a non-perturbative description of colliding electrons*. Nuclear Physics B **925**, 455–469, (2017).
- [Fr73] J. Fröhlich. *On the infrared problem in a model of scalar electrons and massless, scalar bosons*. Ann. Inst. H. Poincaré Sect. A (N.S.) **19**, (1973) 1–103.
- [Fr74.1] J. Fröhlich. *Existence of dressed one electron states in a class of persistent models*. Fortschr. Phys. **22**, (1974) 158–198.
- [FMS79] J. Fröhlich, G. Morchio and F. Strocchi. *Infrared problem and spontaneous breaking of the Lorentz group in QED*. Phys. Lett. B **89**, (1979) 61–64.

- [FP10] J. Fröhlich and A. Pizzo. *Renormalized electron mass in non-relativistic QED*. Commun. Math. Phys. **294**, (2010) 439–470.
- [Ha18] V. Hartenstein. *On the Maxwell-Lorentz dynamics of point charges*. PhD thesis, Ludwig-Maximilians-Universität München, 2018.
- [He17] A. Herdegen. *Asymptotic structure of electrodynamics revisited*. Lett. Mat. Phys. **107**, (2017) 1439–1470.
- [Ho18] Duc Viet Hoang. *Soliton solutions and scattering theory of Maxwell-Newton equations*. Bachelor thesis, Ludwig-Maximilians-Universität München, 2018.
- [KS00] A. Komech and H. Spohn. *Long-time asymptotics for coupled Maxwell-Lorentz equations*. Comm. PDE **25**, (2000) 559–584.
- [KM13] M. Könnenberg and O. Matte. *The mass shell in the semi-relativistic Pauli-Fierz model*. Ann. Henri Poincaré **15**, (2015) 863–915
- [KF70] P. P. Kulish and L. D. Faddeev. *Asymptotic conditions and infrared divergences in quantum electrodynamics*. Theor. Math. Phys. **5**, (1970) 153–166.
- [Pr18] K. Prabhu. *Conservation of asymptotic charges from past to future null infinity: Maxwell fields*. JHEP (2018) 113.
- [RS2] M. Reed and B. Simon, *Methods of modern mathematical physics II. Fourier analysis, self-adjointness*. Academic Press, 1980.
- [Sp] H. Spohn. *Dynamics of charged particles and their radiation fields*. Cambridge 2004
- [St17] A. Strominger. *Lectures on the infrared structure of gravity and gauge theory*. Princeton University Press, 2018.