Higgs Mechanism and Renormalization Group Flow: Are They Compatible?

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Abstract. Usually the Lagrangian of a model for massive vector bosons is derived in a geometric way by the Higgs mechanism. We investigate whether this geometric structure is maintained under the renormalization group (RG) flow. Using the framework of Epstein-Glaser renormalization, we find that the answer is 'no', if the renormalization mass scale(s) are chosen in a way corresponding to the minimal subtraction scheme. This result is derived for the $U(1)$-Higgs model to 1-loop order. On the other hand we give a model-independent proof that physical consistency, which is a weak form of BRST-invariance of the time-ordered products, is stable under the RG-flow.

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1. Introduction

By the renormalization group (RG) flow we have a tool to describe a QFT-model at different scales. In this description, the basic fields, the gauge-fixing parameter, the masses and the prefactors of the various interaction terms are scale-dependent quantities.

On the other hand the derivation of the Lagrangian of a model for massive vector bosons by the Higgs mechanism, i.e. by spontaneous symmetry breaking of a gauge theory, implies that the prefactors of the various interaction terms are uniquely determined functions of the coupling constant(s) and masses.

Do these functions remain unchanged under the RG-flow, i.e. under an arbitrary change of scale? This question is a reformulation of the title of this paper. Since the non-trivial contributions to the RG-flow come from loop diagrams and different interaction terms get different loop-corrections, it is uncertain, whether the answer is 'yes'. Or - one can come to the same conclusion by considering the underlying frameworks: the Higgs mechanism is formulated in classical field theory and, to the best of our knowledge, it is not understood in a pure QFT framework; on the other hand, the RG-flow is a pure quantum effect.

Some readers may wonder, whether the Lagrangian of the scaled model describes still a consistent QFT-model, if it is not derivable by the Higgs mechanism? The answer is 'yes', for the following reasons: since Poincaré invariance, relevant discrete symmetries and renormalizability are maintained under the RG-flow, the crucial requirement for consistency of a quantum gauge model is physical consistency (PC) [17, 9]. This is the condition that the free BRST-charge[1] commutes with the "S-matrix" in the adiabatic limit, in order that the latter induces a well-defined operator on the physical subspace. We give a model-independent proof that PC is maintained under the RG-flow (Theorem 3.1).

In the literature we could not find an explicit 'yes' or 'no' to the question in the title. However, some papers silently assume that the answer is 'yes' – see the few examples mentioned in [11, Introduction]. The answer certainly depends on the renormalization scheme.

We work with the definition of the RG-flow given in framework Epstein-Glaser renormalization [12]: since a scaling transformation amounts to a change of the renormalization prescription, it can equivalently be expressed by a renormalization of the interaction – this is an application of the Main Theorem, see [18, 14, 4, 3]. The so defined RG-flow depends on the renormalization scheme via the two possibilities that the scaling transformation may act on the renormalization mass scale(s) or it may not; and this may be different for different Feynman diagrams.

We investigate the question in the title by explicit 1-loop calculations – the technical details are omitted in this paper, they are given in [11]. To minimise the computations, we study the model of one massive vector field, that is, we start the RG-flow with the $U(1)$-Higgs model.

2. Precise formulation of the question

Lagrangian of the initial model. The just mentioned model has one massive vector field A^μ, the corresponding Stückelberg field B, a further real scalar field φ ("Higgs field") and the Fadeev-Popov ghost fields $(u\,,\,\tilde{u})$. The Lagrangian reads

$$L_{\mathrm{total}} \simeq -\frac{1}{4}\,F^2 + \frac{1}{2}\,(D^\mu\Phi)^* D_\mu\Phi - V(\Phi) + L_{\mathrm{gf}} + L_{\mathrm{ghost}}\ , \qquad (2.1)$$

where $\simeq$ means equal up to the addition of terms of type $\partial^a A$, where $|a| \geq 1$ and A is a local field polynomial. In addition we use the notations $F^2 := (\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu)$,

$$\Phi := iB + \frac{m}{\kappa} + \varphi\ , \quad D^\mu := \partial^\mu - i\kappa\, A^\mu \qquad (2.2)$$

[1] That is the charge implementing the BRST-transformation of the asymptotic free fields.

and

$$V(\Phi) := \frac{\kappa^2 m_H^2}{8m^2}\,(\Phi^*\Phi)^2 - \frac{m_H^2}{4}\,(\Phi^*\Phi) + \frac{m_H^2 m^2}{8\kappa^2}\ , \tag{2.3}$$

where κ is the coupling constant and m and m_H are the masses of the A- and φ-field, respectively, as it turns out below in (2.6). The gauge-fixing and ghost Lagrangian are given by

$$L_{\rm gf} := -\frac{\Lambda}{2}\Big(\partial A + \frac{m}{\Lambda}B\Big)^2 \tag{2.4}$$

and

$$L_{\rm ghost} := \partial\tilde{u}\partial u - \frac{m^2}{\Lambda}\,\tilde{u}u - \frac{\kappa\, m}{\Lambda}\,\tilde{u}u\varphi\ , \tag{2.5}$$

respectively, where Λ is the gauge-fixing parameter. The masses of the A- and φ-field are generated by the Higgs mechanism.

In view of perturbation theory we split $L_{\rm total}$ into a free part L_0 (all bilinear terms) and an interacting part L (all tri- and quadrilinear terms):

$$\begin{aligned} L_0 &= -\frac{1}{4}\,F^2 + \frac{m^2}{2}A^2 + \frac{1}{2}(\partial B)^2 - \frac{m^2}{2\Lambda}\,B^2 - \frac{\Lambda}{2}(\partial A)^2 \\ &\quad + \frac{1}{2}(\partial\varphi)^2 - \frac{m_H^2}{2}\,\varphi^2 + \partial\tilde{u}\partial u - \frac{m^2}{\Lambda}\,\tilde{u}u\ , \end{aligned} \tag{2.6}$$

$$\begin{aligned} L &= \kappa\Big(m\,A^2\varphi - \frac{m^2}{\Lambda}\tilde{u}u\varphi + B(A\partial\varphi) - \varphi(A\partial B) - \frac{m_H^2}{2m}\varphi^3 - \frac{m_H^2}{2m}B^2\varphi\Big) \\ &\quad + \kappa^2\Big(\frac{1}{2}A^2(\varphi^2 + B^2) - \frac{m_H^2}{8m^2}\varphi^4 - \frac{m_H^2}{4m^2}\varphi^2 B^2 - \frac{m_H^2}{8m^2}B^4\Big)\ , \end{aligned} \tag{2.7}$$

where $V^2 := V^\mu V_\mu$, $VW := V^\mu W_\mu$ for Lorentz vectors $V, W \in \mathbb{C}^4$.

Remark 2.1. The BRST-transformation is a graded derivation which commutes with partial derivatives and is given on the basic fields by

$$\begin{aligned} s\,A^\mu = \partial^\mu u\ ,\quad s\,B = mu + \kappa\,u\varphi\ ,\quad s\,\varphi = -\kappa\,Bu\ , \\ s\,u = 0\ ,\quad s\,\tilde{u} = -\Lambda\,(\partial A + \frac{m}{\Lambda}\,B)\ . \end{aligned} \tag{2.8}$$

By $s_0 := s|_{\kappa=0}$ we denote its version for the free theory. We point out that L and L_0 are invariant w.r.t. the pertinent BRST-transformation:

$$sL \simeq 0\ , \qquad s_0 L_0 \simeq 0\ , \tag{2.9}$$

where $\simeq$ has the same meaning as above.

Definition of the RG-flow. In view of Epstein-Glaser renormalization [12] we write

$$L = \kappa\,L_1 + \kappa^2\,L_2 \tag{2.10}$$

and introduce an adiabatic switching of the coupling constant by a test function $g \in \mathcal{D}(\mathbb{R}^4)$:

$$L(g) \equiv L^{\mathbf{m}}(g) := \int dx\,\Big(\kappa\,g(x)\,L_1(x) + \big(\kappa\,g(x)\big)^2\,L_2(x)\Big)\ . \tag{2.11}$$

For later purpose we have introduced the upper index $\mathbf{m} := (m, m_H)$.

In the Epstein-Glaser framework the RG-flow is defined by a scaling transformation σ_ρ [14, 4, 3]:

$$\sigma_\rho^{-1}(\phi(x)) = \rho\,\phi(\rho x)\ ,\quad \phi = A^\mu, B, \varphi, u, \tilde{u}\ ,\quad \rho > 0\ , \tag{2.12}$$

and a simultaneous scaling of the masses $\mathbf{m} \mapsto \rho^{-1}\mathbf{m} = (\rho^{-1}m, \rho^{-1}m_H)$; see [4] for the precise definition of σ_ρ. Under this transformation the classical action is invariant (up to a scaling of the switching function g).

In QFT scaling invariance is in general broken in the process of renormalization. To explain this, we introduce the generating functional $S(iL(g))$ of the time-ordered products of $L(g)$, i.e.

$$T_n(L(g)^{\otimes n}) = \frac{d^n}{i^n\, d\eta^n}|_{\eta=0}\, S(i\eta\, L(g)) \quad \text{or generally} \quad T_n = S^{(n)}(0)\ , \tag{2.13}$$

which we construct inductively by Epstein-Glaser renormalization [12]. To define the RG-flow, we need to perform the adiabatic limit

$$\mathbf{S}[L] := \lim_{\varepsilon\downarrow 0} S(iL(g_\varepsilon))\ ,\quad g_\varepsilon(x) := g(\varepsilon x)\ , \tag{2.14}$$

where $g(0) = 1$ is assumed. For a purely massive model and with a suitable (re)normalization of $S(iL(g))$, this limit exists in the strong operator sense. For a rigorous proof of this statement we refer to [12, 13]; in this paper we treat the adiabatic limit on a heuristic level.

The Main Theorem of perturbative renormalization [18, 4, 14] implies that a scaling transformation of $\mathbf{S}[L]$, i.e.

$$\mathbf{S}_{\mathbf{m}}[L^{\mathbf{m}}] \mapsto \sigma_\rho(\mathbf{S}_{\rho^{-1}\mathbf{m}}[\sigma_\rho^{-1}(L^{\mathbf{m}})])\ ,$$

can equivalently be expressed by a renormalization of the interaction $L^{\mathbf{m}} \mapsto z_\rho(L^{\mathbf{m}})$ ("running interaction"), explicitly

$$\sigma_\rho(\mathbf{S}_{\rho^{-1}\mathbf{m}}[\sigma_\rho^{-1}(L^{\mathbf{m}})]) = \mathbf{S}_{\mathbf{m}}[z_\rho(L^{\mathbf{m}})]\ , \tag{2.15}$$

where the lower index $\mathbf{m}$ of $\mathbf{S}_{\mathbf{m}}$ denotes the masses of the Feynman propagators. This is explained in detail in Sect. 3.

The form of the running interaction. Using general properties of the running interaction (derived in [4]), we know that each term appearing in $z_\rho(L)$ is Lorentz invariant, has ghost number $= 0$ and has mass dimension ≤ 4. In addition, using that L (2.7) is even under the field parity transformation

$$(A, B, \varphi, u, \tilde{u}) \mapsto (-A, -B, \varphi, u, \tilde{u})\ , \tag{2.16}$$

one easily derives that also $z_\rho(L)$ is even under this transformation. One can also show that only one term containing the Fadeev-Popov ghosts can appear in $(z_\rho(L) - L)$, namely a term $\sim \tilde{u}u$. Moreover, with a slight restriction on the (re)normalization of $S(iL(g))$, one can exclude 1-leg terms from $z_\rho(L)$ [11].

Using these facts, we conclude that the running interaction has the form

$$\begin{aligned} z_\rho(L) \simeq \hbar^{-1}\Big[& k_\rho - \frac{1}{4}a_{0\rho}\, F^2 + \frac{m^2}{2}a_{1\rho}A^2 - \frac{a_{2\rho}}{2}(\partial A)^2 + \frac{1}{2}b_{0\rho}(\partial B)^2 - \frac{m^2}{2\,\Lambda}\, b_{1\rho}\, B^2 \\ & + \frac{1}{2}c_{0\rho}\, (\partial\varphi)^2 - \frac{m_H^2}{2}c_{1\rho}\, \varphi^2 - \frac{m^2}{\Lambda}\, c_{2\rho}\, \tilde{u}u + b_{2\rho}\, m\, (A\partial B) \\ & + \kappa\Big((1+l_{0\rho})\, m\, A^2\varphi - \frac{m}{\Lambda}\, \tilde{u}u\varphi + (1+l_{1\rho})\, B(A\partial\varphi) \\ & - (1+l_{2\rho})\, \varphi(A\partial B) - \frac{(1+l_{3\rho})m_H^2}{2m}\, \varphi^3 - \frac{(1+l_{4\rho})m_H^2}{2m}\, B^2\varphi\Big) \\ & + \kappa^2\Big(\frac{(1+l_{5\rho})}{2}A^2\, \varphi^2 + \frac{(1+l_{6\rho})}{2}A^2\, B^2 - \frac{(1+l_{7\rho})m_H^2}{8m^2}\, \varphi^4 \\ & - \frac{(1+l_{8\rho})m_H^2}{4m^2}\, \varphi^2 B^2 - \frac{(1+l_{9\rho})m_H^2}{8m^2}\, B^4 + l_{11\rho}\, (A^2)^2\Big)\Big] \ , \end{aligned} \tag{2.17}$$

where $\simeq$ has the same meaning as in (2.1) and $k_\rho \in \hbar\,\mathbb{C}[[\hbar]]$ is a constant field (it is the contribution of the vacuum diagrams), which may be neglected.

The dimensionless, ρ-dependent coefficients k_ρ, $a_{j\rho}$, $b_{j\rho}$, $c_{j\rho}$ and $l_{j\rho}$ will collectively be denoted by e_ρ. In principle these coefficients are computable – at least to lowest orders; however, at the present stage they are unknown. As shown in [11], the e_ρ's are formal power series in $\kappa^2\hbar$ with vanishing term of zeroth order,

$$e_\rho = \sum_{n=1}^{\infty} e_\rho^{(n)}\, (\kappa^2\hbar)^n \ , \quad e = k, a_j,\, b_j,\, c_j,\, l_j \ . \tag{2.18}$$

Due to $z_{\rho=1}(L) = L/\hbar$, all functions $\rho \mapsto e_\rho$ have the initial value 0 at $\rho = 1$.

Renormalization of the wave functions, masses, gauge-fixing parameter and coupling parameters. Except for the $A\partial B$- and A^4-term, all field monomials appearing in $z_\rho(L)$ are already present in $L_0 + L$. Therefore, introducing new fields, which are of the form

$$\phi_\rho(x) = f_\phi(\rho)\, \phi(x) \ , \quad \phi = A,\, B,\, \varphi, \tag{2.19}$$

where $f_\phi : (0,\infty) \to \mathbb{C}$ is a ϕ-dependent function, and introducing a running gauge-fixing parameter Λ_ρ, running masses $\mathbf{m}_\rho \equiv (m_\rho,\, m_{B\rho},\, m_{u\rho},\, m_{H\rho})$ and running coupling constants $\kappa_\rho\lambda_{j\rho}$, we can achieve that $L_0 + z_\rho(L) - k_\rho$ has roughly the same form as $L_0 + L$:

$$L_0 + z_\rho(L) - k_\rho = L_0^\rho + L^\rho \ , \tag{2.20}$$

where

$$\begin{aligned} L_0^\rho = & -\frac{1}{4}\, F_\rho^2 + \frac{m_\rho^2}{2}A_\rho^2 + \frac{1}{2}(\partial B_\rho)^2 - \frac{m_{B\rho}^2}{2}\, B_\rho^2 - \frac{\Lambda_\rho}{2}(\partial A_\rho)^2 \\ & + \frac{1}{2}(\partial\varphi_\rho)^2 - \frac{m_{H\rho}^2}{2}\, \varphi_\rho^2 + \partial\tilde{u}\partial u - m_{u\rho}^2\, \tilde{u}u \ , \end{aligned} \tag{2.21}$$

(with $F^{\mu\nu}_\rho := \partial^\mu A^\nu_\rho - \partial^\nu A^\mu_\rho$) and

$$\begin{aligned}L^\rho = \kappa_\rho\Big(& m_\rho A^2_\rho \varphi_\rho - \frac{\lambda_{10\rho}\, m^2_{u\rho}}{m_\rho}\tilde{u}u\varphi + \lambda_{1\rho} B_\rho (A_\rho \partial \varphi_\rho)\\ & - \lambda_{2\rho}\varphi_\rho (A_\rho \partial B_\rho) - \frac{\lambda_{3\rho} m^2_{H\rho}}{2m_\rho}\varphi^3_\rho - \frac{\lambda_{4\rho} m^2_{H\rho}}{2m_\rho} B^2_\rho \varphi_\rho\Big)\\ & + \kappa^2\Big(\frac{\lambda_{5\rho}}{2} A^2_\rho \varphi^2_\rho + \frac{\lambda_{6\rho}}{2} A^2_\rho B^2_\rho - \frac{\lambda_{7\rho} m^2_{H\rho}}{8m^2_\rho}\varphi^4_\rho\\ & - \frac{\lambda_{8\rho} m^2_{H\rho}}{4m^2_\rho}\varphi^2_\rho B^2_\rho - \frac{\lambda_{9\rho} m^2_{H\rho}}{8m^2_\rho} B^4_\rho + \lambda_{11\rho}\, A^2_\rho\Big)\\ & + \big((\lambda_{12\rho} - 1) m_\rho + \sqrt{\Lambda_\rho}\, m_{B\rho}\big)\, A_\rho \partial B_\rho\ . \end{aligned} \tag{2.22}$$

In view of the Higgs mechanism for $L + z_\rho(L)$ (2.27), the definition of $\lambda_{12\rho}$ is rather complicated. Apart from the $A\partial B$-term, we have absorbed the novel bilinear interaction terms in the free Lagrangian. Since every new field is of the form (2.19), the condition (2.20) is an equation for polynomials in the old fields; equating the coefficients we obtain the following explicit formulas for the running quantities:

- for the wave functions

$$A^\mu_\rho = \sqrt{1 + a_{0\rho}}\, A^\mu\ , \quad B_\rho = \sqrt{1 + b_{0\rho}}\, B\ , \quad \varphi_\rho = \sqrt{1 + c_{0\rho}}\, \varphi\ ; \tag{2.23}$$

- for the gauge-fixing parameter

$$\Lambda_\rho = \frac{\Lambda + a_{2\rho}}{1 + a_{0\rho}}\ ; \tag{2.24}$$

- for the masses

$$\begin{aligned} m_\rho &= \sqrt{\frac{1 + a_{1\rho}}{1 + a_{0\rho}}}\, m\ , \qquad m_{H\rho} = \sqrt{\frac{1 + c_{1\rho}}{1 + c_{0\rho}}}\, m_H\ ,\\ m_{B\rho} &= \sqrt{\frac{1 + b_{1\rho}}{1 + b_{0\rho}}}\, \frac{m}{\sqrt{\Lambda}}\ , \qquad m_{u\rho} = \sqrt{1 + c_{2\rho}}\, \frac{m}{\sqrt{\Lambda}}\ ; \end{aligned} \tag{2.25}$$

- for the coupling constant

$$\kappa_\rho = \frac{1 + l_{0\rho}}{\sqrt{(1 + a_{0\rho})(1 + a_{1\rho})(1 + c_{0\rho})}}\, \kappa\ ; \tag{2.26}$$

and the running coupling parameters $\lambda_{j\rho}$ are determined analogously.

By the renormalization of the wave functions, masses and gauge fixing-parameter, we change the splitting of the total Lagrangian $L_0 + z_\rho(L)$ into a free and interacting part, i.e. we change the starting point for the perturbative expansion. To justify this, one has to show that the two pertubative QFTs given by the splittings $L_0 + z_\rho(L)$ and $L^\rho_0 + L^\rho$, respectively, have the same physical content.[2] Using the framework of algebraic QFT, one has to show the

[2] This statement can be viewed as an application of the "Principle of Perturbative Agreement" of Hollands and Wald [15].

following: given a renormalization prescription for $L_0 + z_\rho(L)$, there exists a renormalization prescription for $L_0^\rho + L^\rho$, such that, in the algebraic adiabatic limit, the pertinent nets of local observables (see [2] or [4, 3]) are equivalent. This task is beyond the scope of this paper.

Higgs mechanism at an arbitrary scale. Our main question is whether the Lagrangian $L_0^\rho + L^\rho$ can also be derived by the Higgs mechanism for all $\rho > 0$. By the latter we mean

$$L_0^\rho + L^\rho \simeq -\frac{1}{4} F_\rho^2 + \frac{1}{2} (D_\rho^\mu \Phi_\rho)^* D_{\rho\mu} \Phi_\rho - V_\rho(\Phi_\rho) + L_{\rm gf}^\rho + L_{\rm ghost}^\rho \ , \tag{2.27}$$

where Φ_ρ, D_ρ and $V_\rho(\Phi_\rho)$ are obtained from (2.2)-(2.3) by replacing $(A^\mu,\, B,\, \varphi,\, m,\, m_H,\, \Lambda)$ by $(A_\rho^\mu,\, B_\rho,\, \varphi_\rho,\, m_\rho,\, m_{H\rho},\, \Lambda_\rho)$ and

$$\begin{aligned} L_{\rm gf}^\rho &:= -\frac{\Lambda_\rho}{2} \left(\partial A_\rho + \frac{m_{B\rho}}{\sqrt{\Lambda_\rho}} B_\rho \right)^2 \ , \\ L_{\rm ghost}^\rho &:= \partial \tilde{u} \cdot \partial u - m_{u\rho}^2\, \tilde{u} u - \frac{\kappa_\rho\, \lambda_{10\rho}\, m_{u\rho}^2}{m_\rho}\, \tilde{u} u \varphi_\rho \ . \end{aligned} \tag{2.28}$$

For the property (2.27) we also say that the model "can be geometrically interpreted as a spontaneously broken gauge theory at all scales" [11]. By a straightforward calculation we find that (2.27) is equivalent to

$$\lambda_{1\rho} = \lambda_{2\rho} = ... = \lambda_{9\rho} = 1 \ , \quad \lambda_{11\rho} = \lambda_{12\rho} = 0 \ . \tag{2.29}$$

To simplify the calculations we assume that initially we are in Feynman gauge: $\Lambda_{\rho=1} = 1$. With that the geometrical interpretability (2.29) is equivalent to the following relations among the coefficients e_ρ:

$$\lambda_{1\rho} = 1 \ \text{gives} \qquad \frac{1 + l_{1\rho}}{1 + l_{0\rho}} = \sqrt{\frac{1 + b_{0\rho}}{1 + a_{1\rho}}} \ , \tag{2.30}$$

$$\lambda_{2\rho} = 1 \ \text{gives} \qquad l_{2\rho} = l_{1\rho} \ , \tag{2.31}$$

$$\lambda_{3\rho} = 1 \ \text{gives} \qquad \frac{1 + l_{3\rho}}{1 + l_{0\rho}} = \frac{1 + c_{1\rho}}{1 + a_{1\rho}} \ , \tag{2.32}$$

$$\lambda_{4\rho} = 1 \ \text{gives} \qquad \frac{1 + l_{4\rho}}{1 + l_{3\rho}} = \frac{1 + b_{0\rho}}{1 + c_{0\rho}} \ , \tag{2.33}$$

$$\lambda_{5\rho} = 1 \ \text{gives} \qquad \frac{1 + l_{5\rho}}{(1 + l_{0\rho})^2} = \frac{1}{1 + a_{1\rho}} \ , \tag{2.34}$$

$$\lambda_{6\rho} = 1 \ \text{gives} \qquad \frac{1 + l_{6\rho}}{1 + l_{5\rho}} = \frac{1 + b_{0\rho}}{1 + c_{0\rho}} \ , \tag{2.35}$$

$$\lambda_{7\rho} = 1 \text{ gives} \qquad \frac{1+l_{7\rho}}{(1+l_{0\rho})^2} = \frac{1+c_{1\rho}}{(1+a_{1\rho})^2} \ , \tag{2.36}$$

$$\lambda_{8\rho} = 1 \text{ gives} \qquad \frac{1+l_{8\rho}}{1+l_{7\rho}} = \frac{1+b_{0\rho}}{1+c_{0\rho}} \ , \tag{2.37}$$

$$\lambda_{9\rho} = 1 \text{ gives} \qquad \frac{1+l_{9\rho}}{1+l_{7\rho}} = \Big(\frac{1+b_{0\rho}}{1+c_{0\rho}}\Big)^2 \ , \tag{2.38}$$

$$\lambda_{11\rho} = 0 \text{ gives} \qquad l_{11\rho} = 0 \ , \tag{2.39}$$

$$\lambda_{12\rho} = 0 \text{ gives} \qquad b_{2\rho} = \sqrt{(1+a_{2\rho})(1+b_{1\rho})} - \sqrt{(1+a_{1\rho})(1+b_{0\rho})} \ . \tag{2.40}$$

Combining the equations (2.32), (2.34) and (2.36) we obtain

$$\frac{1+l_{7\rho}}{1+l_{3\rho}} = \frac{1+l_{5\rho}}{1+l_{0\rho}} \ . \tag{2.41}$$

This condition and (2.40) are crucial for the geometrical interpretability, as we will see.

Remark 2.2. BRST-invariance of $L_0 + z_\rho(L)$ is a clearly stronger property than the geometrical interpretability (2.27). More precisely: considering the coefficients e_ρ as unknown and assuming that $s(L_0 + z_\rho(L)) \simeq 0$, we obtain rather restrictive relations among the coefficients e_ρ which imply the equations (2.30)-(2.40). Ignoring k_ρ, the number of coefficients e_ρ, which are left freely chosable by the BRST-property, is 3; and for the geometrical interpretability this number is 9 – see [11].

3. Physical consistency and perturbative gauge invariance

Physical consistency (PC). The generic problem of a model containing spin 1 fields, is the presence of unphysical fields. A way to solve this problem in a scattering framework is to construct $S(iL(g))$ such that the following holds. For the asymptotic free fields let $\mathcal{H}_{\mathrm{phys}}$ be the "subspace" of physical states. In the adiabatic limit lim $g \to 1$, $S(iL(g))$ has to induce a well defined operator from $\mathcal{H}_{\mathrm{phys}}$ into itself, which is the physically relevant S-matrix.

To formulate this condition explicitly, let Q be the generator of the free BRST-transformation $s_0 := s|_{\kappa=0}$:

$$[Q,\phi]_\star^\mp \approx i\hbar\, s_0\phi \ , \quad \phi = A^\mu \, , B \, , \varphi \, , u \, , \tilde{u} \ , \tag{3.1}$$

where $[\cdot\, , \cdot]_\star^\mp$ denotes the graded commutator w.r.t. the $\star$-product and $\approx$ means 'equal modulo the free field equations'. With that we may write $\mathcal{H}_{\mathrm{phys}} := \frac{\ker Q}{\mathrm{ran}\, Q}$, and the mentioned, fundamental condition on $S(iL(g))$ is equivalent to

$$0 \approx [Q, \mathbf{S}[L]]_\star|_{\ker Q} \equiv \lim_{\varepsilon\downarrow 0}[Q, S(iL(g_\varepsilon)/\hbar)]_\star|_{\ker Q} \ , \tag{3.2}$$

see [17, 9]. For simplicity we omit the restriction to $\ker Q$ and call the resulting condition "physical consistency (PC)".

Stability of PC under the RG-flow. A main, model-independent result of this paper is that PC is maintained under the RG-flow.

Theorem 3.1. *Assume that* $S_{\mathbf{m}}(iL(g))$ *is renormalized such that the adiabatic limit* $\varepsilon \downarrow 0$ *exists and is unique for* $\sigma_\rho \circ S_{\rho^{-1}\mathbf{m}} \circ \sigma_\rho^{-1}(iL(g_\varepsilon))\ \ \forall \rho > 0$*, and such that* $S_{\mathbf{m}}(iL^{\mathbf{m}}(g))$ *fulfills PC for all values* $m_j > 0$ *of the masses* $\mathbf{m} = (m_j)$*. Then, the following holds:*

$$\big[Q, \mathbf{S}[z_\rho(L)]\big]_\star \equiv \lim_{\varepsilon\downarrow 0}[Q, S(iz_\rho(L)(g_\varepsilon)]_\star \approx 0\ , \quad \forall \rho > 0\ . \tag{3.3}$$

Hence, at least in this weak form, BRST-invariance of the time-ordered products is stable under the RG-flow.

Proof. As a preparation we explain the construction of $z_\rho(L)$ and derive (2.15). Assuming that S fulfills the axioms of Epstein-Glaser renormalization, this holds also for the scaled time-ordered products $\sigma_\rho \circ S \circ \sigma_\rho^{-1}$; therefore, the Main Theorem [4, 14] applies: there exists a unique map $Z_\rho \equiv Z_{\rho,\mathbf{m}}$ from the space of local interactions into itself such that

$$\sigma_\rho \circ S_{\rho^{-1}\mathbf{m}} \circ \sigma_\rho^{-1} = S_{\mathbf{m}} \circ Z_{\rho,\mathbf{m}} \tag{3.4}$$

(the lower index $\mathbf{m}$ on S and Z_ρ denotes the masses of the underlying $\star$-product, i.e. the masses of the Feynman propagators).

In view of the adiabatic limit we investigate $Z_\rho(iL(g_\varepsilon)/\hbar)$ and take into account that $\partial g_\varepsilon(x) \sim \varepsilon$. From [4, Prop. 4.3] we know that there exist local field polynomials $p_{k\rho}(L)$ such that

$$Z_\rho(iL(g_\varepsilon)/\hbar) = \frac{i}{\hbar}\Big(L(g_\varepsilon) + \sum_{k=2}^{\infty}\int dx\ p_{k\rho}(L)(x)\,(\kappa g_\varepsilon(x))^k\Big) + \mathcal{O}(\varepsilon)\ . \tag{3.5}$$

Obviously, $p_{k\rho}(L)$ is not uniquely determined: one may add terms of type $\partial^a A$, $|a| \geq 1$, where A is a local field polynomial. Setting

$$z_\rho(L)(g) := \frac{1}{\hbar}\sum_{k=1}^{\infty}\int dx\ \Big(L_k(x) + p_{k\rho}(L)(x)\Big)\,(\kappa g(x))^k\ , \tag{3.6}$$

where $p_{1\rho} := 0$ and $L_k := 0$ for $k \geq 3$, we obtain

$$Z_\rho(iL(g_\varepsilon)/\hbar) = i\, z_\rho(L)(g_\varepsilon) + \mathcal{O}(\varepsilon)\ . \tag{3.7}$$

Using this result and (multi-)linearity of the time-ordered products, we obtain (2.15):

$$\begin{aligned}\sigma_\rho(\mathbf{S}_{\rho^{-1}\mathbf{m}}[\sigma_\rho^{-1}(L^{\mathbf{m}})]) &:= \lim_{\varepsilon\downarrow 0}\sigma_\rho \circ S_{\rho^{-1}\mathbf{m}} \circ \sigma_\rho^{-1}(iL^{\mathbf{m}}(g_\varepsilon))\\ &= \lim_{\varepsilon\downarrow 0} S_{\mathbf{m}}\big(Z_\rho(iL^{\mathbf{m}}(g_\varepsilon))\big) = \lim_{\varepsilon\downarrow 0} S_{\mathbf{m}}\big(i\, z_\rho(L^{\mathbf{m}})(g_\varepsilon)\big) =: \mathbf{S}_{\mathbf{m}}[z_\rho(L^{\mathbf{m}})]\ .\end{aligned} \tag{3.8}$$

By assumption the limit exists on the l.h.s.; hence, it exists also on the r.h.s..

With these tools we are able to prove (3.3): using the relations

$$\sigma_\rho^{-1}(L^{\mathbf{m}}(g)) = L^{\rho^{-1}\mathbf{m}}(g_{1/\rho}) \qquad (\text{again } g_\lambda(x) := g(\lambda x)) \tag{3.9}$$

and

$$\sigma_\rho(F \star_{\rho^{-1}\mathbf{m}} G) = \sigma_\rho(F) \star_{\mathbf{m}} \sigma_\rho(G) \ , \quad \rho\, \sigma_\rho \circ Q_{\rho^{-1}\mathbf{m}} = Q_{\mathbf{m}} \ , \tag{3.10}$$

we obtain

$$\begin{aligned}[][Q_{\mathbf{m}}, S_{\mathbf{m}}(Z_\rho(iL^{\mathbf{m}}(g_\varepsilon)))]_{\star_{\mathbf{m}}} &= [Q_{\mathbf{m}}, \sigma_\rho \circ S_{\rho^{-1}\mathbf{m}}(iL^{\rho^{-1}\mathbf{m}}(g_{\varepsilon/\rho}))]_{\star_{\mathbf{m}}} \\ &= \rho\, \sigma_\rho\Big([Q_{\rho^{-1}\mathbf{m}}, S_{\rho^{-1}\mathbf{m}}(iL^{\rho^{-1}\mathbf{m}}(g_{\varepsilon/\rho}))]_{\star_{\rho^{-1}\mathbf{m}}}\Big) \ . \end{aligned} \tag{3.11}$$

By assumption, the adiabatic limit $\varepsilon \downarrow 0$ vanishes for the last expression. (Due to uniqueness of the adiabatic limit, it does not matter whether we perform this limit with g or $g_{1/\rho}$.) With that and with (3.7) we conclude

$$0 \approx \lim_{\varepsilon\downarrow 0}\, [Q, S(Z_\rho(iL(g_\varepsilon)))]_\star = \lim_{\varepsilon\downarrow 0}\, [Q, S(i\, z_\rho(L)(g_\varepsilon))]_\star = \big[Q, \mathbf{S}[z_\rho(L)]\big]_\star \ .$$

□

Perturbative gauge invariance (PGI). For the initial model $S(iL(g))$ we admit all renormalization prescriptions which fulfill the Epstein-Glaser axioms [12, 4] and *perturbative gauge invariance (PGI)* [7, 8, 19, 6]. The latter is a somewhat stronger version of PC, which is formulated *before* the adiabatic limit $g \to 1$ is taken.

In detail, PGI is the condition that to the given interaction $L(g)$ (2.11) there exists a "Q-vertex"

$$\mathcal{P}^\nu(g; f) := \int dx\, \Big(\kappa\, P_1^\nu(x) + \kappa^2 g(x)\, P_2^\nu(x)\Big)\, f(x), \tag{3.12}$$

(where $g, f \in \mathcal{D}(\mathbb{R}^4)$ and P_1, P_2 are local field polynomials) and a renormalization of the time-ordered products such that

$$[Q, S\big(i\, L(g)\big)]_\star \approx \frac{d}{d\eta}|_{\eta=0}\, S\big(i\, L(g) + \eta\, \mathcal{P}^\nu(g; \partial_\nu g)\big) \ . \tag{3.13}$$

The latter equation is understood in the sense of formal power series in κ and $\hbar$.

That PGI implies PC, is easy to see (on the heuristic level on which we treat the adiabatic limit in this paper): the r.h.s. of (3.13) vanishes in the adiabatic limit, since it is linear in the Q-vertex, the latter is linear in $\partial_\nu g$ and $\partial_\nu g_\varepsilon \sim \varepsilon$.

Requiring PGI, renormalizability and some obvious properties as Poincaré invariance and relevant discrete symmetries, the Lagrangian of the Standard model of electroweak interactions has been derived in [8, 1]. In this way the presence of Higgs particles and chirality of fermionic interactions can be understood without recourse to any geometrical or group theoretical concepts (see also [21]).

It is well-known that the $U(1)$-Higgs model is anomaly-free. Hence, our initial model can be renormalized such that PGI (3.13) holds true for all values of $m, m_H > 0$. Using Theorem 3.1, we conclude that this model is consistent at all scales.

4. Higgs mechanism at all scales to 1-loop order

In this section we explain, how one can fulfill the validity of the Higgs mechanism at all scales, i.e. the equations (2.30)-(2.40), on 1-loop level.

4.1. The two ways to renormalize

To write the fundamental formula (3.4) to n-th order, we use the chain rule:

$$Z_{\rho,\mathbf{m}}^{(n)}\big(L(g)^{\otimes n}\big) = \sigma_\rho \circ T_{n\,\mathbf{m}/\rho}\big((\sigma_\rho^{-1}\, L(g))^{\otimes n}\big) - T_{n\,\mathbf{m}}\big(L(g)^{\otimes n}\big)$$
$$- \sum_{P\in\mathrm{Part}(\{1,\dots,n\}),\, n>|P|>1} T_{|P|\,\mathbf{m}}\big(\otimes_{I\in P} Z_{\rho,\mathbf{m}}^{|I|}(L(g)^{\otimes |I|})\big)\ , \qquad (4.1)$$

where $Z_\rho^{(n)} := Z_\rho^{(n)}(0)$ is the n-th derivative of $Z_\rho(F)$ at $F=0$ and the two terms with $|P|=n$ and $|P|=1$, resp., are explicitly written out.

We are now going to investigate the contribution to the r.h.s. of (4.1) of a primitive divergent diagram Γ, i.e. Γ has singular order[3] $\omega(\Gamma)\geq 0$ and does not contain any subdiagram $\Gamma_1\subset\Gamma$ with less vertices and with $\omega(\Gamma_1)\geq 0$. For such a diagram, the expression in the second line of (4.1) vanishes.

Denoting the contribution of Γ to $T_{n\,\mathbf{m}}\big(L(g)^{\otimes n}\big)$ by

$$\int dx_1\dots dx_n\ t^\Gamma_{\mathbf{m}}(x_1-x_n,\dots,x_{n-1}-x_n)\, P^\Gamma(x_1,\dots,x_n)\ \prod_{k=1}^n (\kappa g(x_k))^{j_k}$$

(where $P^\Gamma(x_1,\dots,x_n)$ is a, in general non-local, field monomial and the values of $j_1,\dots,j_n\in\{1,2\}$ depend on Γ), the computation of the contribution of Γ to $Z_{\rho,\mathbf{m}}^{(n)}\big(L(g)^{\otimes n}\big)$ amounts to the computation of

$$\rho^{D^\Gamma}\ t^\Gamma_{\mathbf{m}/\rho}(\rho y) - t^\Gamma_{\mathbf{m}}(y)\ , \qquad (4.2)$$

where $D^\Gamma := \omega(\Gamma)+4(n-1)\in\mathbb{N}$ and $y:=(x_1-x_n,\dots,x_{n-1}-x_n)$.

For simplicity we assume that $0\leq\omega(\Gamma)<2$; this assumption is satisfied for all 1-loop calculations which are done in [11] and whose results are used in this paper. Applying the scaling and mass expansion ("sm-expansion") [10], we then know that $t^\Gamma_{\mathbf{m}}$ is of the form

$$t^\Gamma_{\mathbf{m}}(y) = t^\Gamma(y) + r^\Gamma_{\mathbf{m}}(y)\ ,\quad r^\Gamma_{\mathbf{m}} = \mathcal{O}(\mathbf{m}^2)\ ,\quad \omega(r^\Gamma_{\mathbf{m}})<0\ , \qquad (4.3)$$

where $t^\Gamma := t^\Gamma_{\mathbf{m}=\mathbf{0}}$ (i.e. all Feynman propagators are replaced by their massless version). The remainder scales homogeneously, $\rho^{D^\Gamma}\, r^\Gamma_{\mathbf{m}/\rho}(\rho y) = r^\Gamma_{\mathbf{m}}(y)$, because it can be renormalized by direct extension (see footnote 3).

To investigate $\rho^{D^\Gamma}\, t^\Gamma(\rho y) - t^\Gamma(y)$, we omit the upper index Γ and use the notations $\omega:=\omega(\Gamma)$, $l:=(n-1)$ and $Y_j := y_j^2 - i0$. We start with the

[3] For $t\in\mathcal{D}'(\mathbb{R}^l)$ or $t\in\mathcal{D}'(\mathbb{R}^l\setminus\{0\})$, the singular order is defined as $\omega(t):=\mathrm{sd}(t)-l$, where $\mathrm{sd}(t)$ is Steinmann's scaling degree of t, which measures the UV-behaviour of t [20]. In the Epstein-Glaser framework, renormalization is the extension of a distribution $t^\circ\in\mathcal{D}'(\mathbb{R}^l\setminus\{0\})$ to a distribution $t\in\mathcal{D}'(\mathbb{R}^l)$, with the condition that $\mathrm{sd}(t)=\mathrm{sd}(t^\circ)$. In the case $\mathrm{sd}(t^\circ)<l$, the extension is unique, due to the scaling degree requirement, and obtained by "direct extension", see [2, Theorem 5.2], [4, Appendix B] and [5, Theorem 4.1].

unrenormalized version $t^\circ \in \mathcal{D}'(\mathbb{R}^{4l}\setminus\{0\})$ of $t := t^\Gamma$, which scales homogeneously:

$$\rho^{\omega+4l}\, t^\circ(\rho y) = t^\circ(y)\ . \tag{4.4}$$

We work with an analytic regularization [16]:

$$t^{\zeta\circ}(y) := t^\circ(y)\,(M^{2l}Y_1\dots Y_l)^\zeta\ , \tag{4.5}$$

where $\zeta\in\mathbb{C}\setminus\{0\}$ with $|\zeta|$ sufficiently small, and $M>0$ is a renormalization mass scale. $t^{\zeta\circ}$ scales also homogeneously – by the regularization we gain that the degree (of the scaling) is $(\omega+4l-2l\zeta)$, which is not an integer. Therefore, the homogeneous extension $t^\zeta\in\mathcal{D}'(\mathbb{R}^{4l})$ is unique and can explicitly be written down by differential renormalization [5, Sect. IV.D].

Using minimal subtraction for the limit $\zeta\to 0$ we obtain an admissible extension $t^M\in\mathcal{D}'(\mathbb{R}^{4l})$ of t° [5, Corollary 4.4]:

$$t^M(y) = \frac{(-1)^\omega}{\omega!}\sum_{r_1\dots r_{\omega+1}}\partial_{y_{r_{\omega+1}}}\dots\partial_{y_{r_1}}\Big[\frac{1}{2l}\Big(\overline{y_{r_1}\dots y_{r_{\omega+1}}\,t^\circ(y)\,\log(M^{2l}Y_1\dots Y_l)}\Big) + \Big(\sum_{j=1}^{\omega}\frac{1}{j}\Big)\Big(\overline{y_{r_1}\dots y_{r_{\omega+1}}\,t^\circ(y)}\Big)\Big]\ , \tag{4.6}$$

where $\sum_r \partial_{y_r}(y_r\dots) := \sum_r \partial^{y_r}_\mu(y^\mu_r\dots)$ and the overline denotes the direct extension. By means of (4.3) we obtain the corresponding distribution of the massive model: $t^M_{\mathbf{m}} := t^M(y)+r_{\mathbf{m}}$. In the following we use that

$$\frac{(-1)^\omega}{\omega!}\sum_{r_1\dots r_{\omega+1}}\partial_{y_{r_{\omega+1}}}\dots\partial_{y_{r_1}}\Big(\overline{y_{r_1}\dots y_{r_{\omega+1}}\,t^\circ(y)}\Big) = \sum_{|a|=\omega} C_a\,\partial^a\delta(y)$$

for some M-independent numbers $C_a\in\mathbb{C}$, as explained after formula (104) in [5].

Whether the expression (4.2) vanishes depends on the following choice:

(A) if we choose for M a fixed mass scale, which is independent of m, m_H, homogeneous scaling is broken:

$$\rho^{\omega+4l}\,t^M_{\mathbf{m}/\rho}(\rho y) - t^M_{\mathbf{m}}(y) = \rho^{\omega+4l}\,t^M(\rho y) - t^M(y) = \log\rho\sum_{|a|=\omega} C_a\,\partial^a\delta(y)\ , \tag{4.7}$$

The breaking term is unique, i.e. independent of M; therefore, we may admit different values of M for different diagrams, however, all M's must be independent of m, m_H.

(B) In contrast, choosing M such that it is subject to our scaling transformation, i.e. $M := \alpha_1 m + \alpha_2 m_H$ where $(\alpha_1,\alpha_2)\in(\mathbb{R}^2\setminus\{(0,0)\})$ may be functions of $\frac{m}{m_H}$, the diagram Γ does not contribute to the RG-flow:

$$\rho^{\omega+4l}\,t^{M/\rho}_{\mathbf{m}/\rho}(\rho y) - t^M_{\mathbf{m}}(y) = \rho^{\omega+4l}\,t^{M/\rho}(\rho y) - t^M(y) = 0\ . \tag{4.8}$$

Remark 4.1. The requirement that the initial $U(1)$-Higgs model fulfills PGI, is neither in conflict with method (A) nor with method (B), for the following reason: we require PGI only for the initial model. Now, working at one fixed scale, the renormalization constant M appearing in (4.6) may have any value $M > 0$ for both methods (A) and (B) and, hence, one may choose it such that PGI is satisfied. These methods only prescribe how M behaves under a scaling transformation: using (A) it remains unchanged, using (B) it is also scaled: $M \mapsto \rho^{-1} M$.

4.2. Equality of certain coefficients to 1-loop order

We explain the basic idea in terms of the two diagrams

$$t^\circ_{1\,\mathbf{m}}(y) := \omega_0\Big(T_2(A^\mu\varphi(x_1)\otimes A^\nu\varphi(x_2))\Big) = -\hbar^2 g^{\mu\nu}\,\Delta^F_m(y)\,\Delta^F_{m_H}(y)\;,$$
$$t^\circ_{2\,\mathbf{m}}(y) := \omega_0\Big(T_2(A^\mu B(x_1)\otimes A^\nu B(x_2))\Big) = -\hbar^2 g^{\mu\nu}\,(\Delta^F_m(y))^2\;,$$

$t^\circ_{1\,\mathbf{m}}, t^\circ_{2\,\mathbf{m}} \in \mathcal{D}'(\mathbb{R}^4\setminus\{0\})$, where ω_0 denotes the vacuum state and $y := x_1 - x_2$. These diagrams are related by the exchange of an inner φ-line with an inner B-line. The essential point is that in the sm-expansion of these two distributions,

$$t^\circ_{j\,\mathbf{m}}(y) = t^\circ_j(y) + r^\circ_{j\,\mathbf{m}}(y)\;,\quad r^\circ_{j\,\mathbf{m}} = \mathcal{O}(\mathbf{m}^2)\;,\quad \omega(r^\circ_{j\,\mathbf{m}}) < 0\;,\quad j = 1,2\;, \tag{4.9}$$

the first term (which is the corresponding massless distribution) is the same: $t^\circ_1(y) = (D^F(y))^2 = t^\circ_2(y)$.

Renormalization is done by extending each term on the r.h.s. of (4.9) individually and by composing these extensions: $t_{j\,\mathbf{m}} := t_j + r_{j\,\mathbf{m}} \in \mathcal{D}'(\mathbb{R}^4)$. For the remainders $r^\circ_{j\,\mathbf{m}}$ the direct extension applies (see footnote 3), which maintains homogeneous scaling: $\rho^4\, r_{j\,\mathbf{m}/\rho}(\rho y) = r_{j\,\mathbf{m}}(y)$. We conclude: if we renormalize t°_1 and t°_2 both by method (A) or both by method (B), we obtain

$$\begin{aligned}\rho^4\, t_{1\,\mathbf{m}/\rho}(\rho y) - t_{1\,\mathbf{m}}(y) &= \rho^4\, t_1(\rho y) - t_1(y)\\ &= \rho^4\, t_2(\rho y) - t_2(y) = \rho^4\, t_{2\,\mathbf{m}/\rho}(\rho y) - t_{2\,\mathbf{m}}(y)\;.\end{aligned}$$

We point out that different renormalization mass scales M for t_1 and t_2 are admitted, only their behaviour under the scaling transformation must be the same. Therefore, this renormalization prescription is compatible with PGI of the initial $U(1)$-Higgs model.

Renormalizing certain Feynman diagrams, which go over into each other by exchanging $B \leftrightarrow \varphi$ for some lines, by the same method (in this sense) – also triangle and square diagrams with derivatives are concerned – we obtain that some of the coefficients e_ρ agree to 1-loop order:

$$c^{(1)}_{0\rho} = b^{(1)}_{0\rho}\;,\;\; l^{(1)}_{1\rho} = l^{(1)}_{2\rho}\;,\;\; l^{(1)}_{3\rho} = l^{(1)}_{4\rho}\;,\;\; l^{(1)}_{5\rho} = l^{(1)}_{6\rho}\;,\;\; l^{(1)}_{7\rho} = l^{(1)}_{8\rho} = l^{(1)}_{9\rho}\;, \tag{4.10}$$

for details see [11]. With that the equations (2.31), (2.33), (2.35) and (2.37)-(2.38) are fulfilled.

In addition, the condition

$$l^{(1)}_{11\rho} = 0\;, \tag{4.11}$$

which is (2.39) to 1-loop order, can be derived from the stability of PC under the RG-flow, by selecting from (3.3) the local terms which are $\sim A^2\, A\partial u$ and by using results of Appendix A in [9].

4.3. Changing the running interaction by finite renormalization

On our way to fulfil the equations (2.30)-(2.40) on 1-loop level, we may use that the following finite renormalizations are admitted by the axioms of causal perturbation theory [12, 4, 10] and that they preserve PGI of the initial model: to $T_2\big(L_1(x_1)\otimes L_1(x_2)\big)$ we may add

$$\hbar^2\,\Big(\alpha_1\,(\partial\varphi)^2(x_1)+\alpha_2\,m_H^2\,\varphi^2(x_1)+\alpha_3\,F^2(x_1)+\alpha_4\,(\partial A+mB)^2 \\ +\alpha_5\,\big(-m^2\,B^2(x_1)+(\partial B)^2(x_1)\big)+\alpha_6\big(m^2\,A^2(x_1)-(\partial A)^2(x_1)\big) \\ +\alpha_7\,m^2\,\big(-2\,\tilde{u}u(x_1)+A^2(x_1)-B^2(x_1)\big)\Big)\,\delta(x_1-x_2)\,\log\tfrac{m}{M}\ , \qquad (4.12)$$

where $\alpha_1,\dots,\alpha_7\in\mathbb{C}$ are arbitrary.

These finite renormalizations modify the 1-loop coefficients $e_\rho^{(1)}$ appearing in $z_\rho(L)$ (2.17) as follows:

$$a_{0\rho}^{(1)}\mapsto a_{0\rho}^{(1)}+2i\,\alpha_3\,\log\rho\ , \qquad (4.13)$$

$$a_{1\rho}^{(1)}\mapsto a_{1\rho}^{(1)}-i\,(\alpha_6+\alpha_7)\,\log\rho\ , \qquad (4.14)$$

$$a_{2\rho}^{(1)}\mapsto a_{2\rho}^{(1)}+i\,(\alpha_4-\alpha_6)\,\log\rho\ , \qquad (4.15)$$

$$b_{0\rho}^{(1)}\mapsto b_{0\rho}^{(1)}-i\,\alpha_5\,\log\rho\ , \qquad (4.16)$$

$$b_{1\rho}^{(1)}\mapsto b_{1\rho}^{(1)}+i\,(\alpha_4-\alpha_5-\alpha_7)\,\log\rho\ , \qquad (4.17)$$

$$b_{2\rho}^{(1)}\mapsto b_{2\rho}^{(1)}+i\,\alpha_4\,\log\rho\ , \qquad (4.18)$$

$$c_{0\rho}^{(1)}\mapsto c_{0\rho}^{(1)}-i\,\alpha_1\,\log\rho\ , \qquad (4.19)$$

$$c_{1\rho}^{(1)}\mapsto c_{1\rho}^{(1)}+i\,\alpha_2\,\log\rho\ , \qquad (4.20)$$

$$c_{2\rho}^{(1)}\mapsto c_{2\rho}^{(1)}-i\,\alpha_7\,\log\rho\ , \qquad (4.21)$$

the other coefficients remain unchanged.

We did not find any further finite renormalizations, which fufill, besides the already mentioned conditions, the requirements
- that they do not add "by hand" novel kind of terms to $(z_\rho(L)-L)$ (see (2.17)) as e.g. terms $\sim\partial\tilde{u}\partial u$ or $\sim m\,\tilde{u}u\varphi$, and
- that the equations (4.10) are preserved.

See [11] for details.

4.4. How to fulfill the Higgs mechanism at all scales

There are two necessary conditions for the Higgs mechanism at all scales, which are crucial, since they cannot be fulfilled by finite renormalizations.

Verification of the first crucial necessary condition. The condition (2.41) reads to 1-loop level

$$l_{7\rho}^{(1)}-l_{3\rho}^{(1)}=l_{5\rho}^{(1)}-l_{0\rho}^{(1)}\ . \qquad (4.22)$$

Since the admissible finite renormalizations (4.12) do not modify the coefficients $l_{j\rho}^{(1)}$, there is no possibility to fulfill (4.22) in this way. However, computing explicitly the relevant coefficients $l_{j\rho}^{(1)}$ by using the renormalization method (A) for all contributing terms, we find that (4.22) holds indeed true. This computation, which is given in [11], involves cancellations of square- and triangle-diagrams – this shows that (4.22) is of a deeper kind than the equalities derived in Sect. 4.2.

The identity (4.22) holds also if certain classes of corresponding diagrams are renormalized by method (B).

How to fulfill the second crucial necessary condition. The condition (2.40) reads to 1-loop order

$$b^{(1)}_{2\rho} = \tfrac{1}{2}\big(a^{(1)}_{2\rho} + b^{(1)}_{1\rho} - a^{(1)}_{1\rho} - b^{(1)}_{0\rho}\big)\ . \tag{4.23}$$

Performing the finite renormalizations (4.12), i.e. inserting (4.13)-(4.21) into (4.23), we find that all α_j drop out – that is, the condition (4.23) cannot be fulfilled by means of these finite renormalizations.

Computing the explicit values for the coefficients $a^{(1)}_{j\rho}$, $b^{(1)}_{j\rho}$ by using method (A) (see [11]), we find that (4.23) does not hold. Hence, using method (A) throughout, we have $\lambda_{12\rho} \neq 0$, i.e. the geometrical interpretation (2.27) is violated by terms $\sim A\partial B$.

However, we can fulfill the condition (4.23) by switching the method from (A) to (B) for all diagrams contributing to $b^{(1)}_{1\rho}$ and a part of the diagrams contributing to $a^{(1)}_{1\rho}$ [11]. (This switch concerns also all diagrams contributing to $a^{(1)}_{0\rho}$, hence we obtain $a^{(1)}_{0\rho} = 0$.)

A family of solutions of the Higgs mechanism at all scales. The conditions (2.30)-(2.40) can be solved to 1-loop order as follows: initially we renormalize all diagrams by using method (A), except for the diagrams just mentioned, for which we use method (B) to fulfill the second crucial necessary condition (4.23). Then we take into account the possibility to modify the coefficients $e^{(1)}_\rho$ by finite renormalizations (4.13)-(4.21). This procedure yields the following family of solutions:

$$\begin{aligned}
&a^{(1)}_{0\rho} = 2\beta_1\, L_\rho\ , \quad a^{(1)}_{1\rho} = -4\, L_\rho\ , \quad a^{(1)}_{2\rho} = (\beta_2 - \beta_3)\, L_\rho\ ,\\
&b^{(1)}_{0\rho} = c^{(1)}_{0\rho} = (2 + 2l_1)\, L_\rho\ , \quad b^{(1)}_{1\rho} = (4 + 2l_1 + \beta_2 + \beta_3)\, L_\rho\ ,\\
&b^{(1)}_{2\rho} = (3 + \beta_2)\, L_\rho\ , \quad c^{(1)}_{1\rho} = -\Big(6\frac{m^2}{m_H^2} + 5\frac{m_H^2}{m^2}\Big)\, L_\rho\ , \quad c^{(1)}_{2\rho} = (-1 + \beta_3)\, L_\rho\ ,
\end{aligned}$$

$$\begin{aligned}
&l^{(1)}_{0\rho} = -3\, L_\rho\ , \quad l^{(1)}_{1\rho} = l^{(1)}_{2\rho} =: l_1\, L_\rho\ , \quad l^{(1)}_{3\rho} = l^{(1)}_{4\rho} = \Big(1 - 6\frac{m^2}{m_H^2} - 5\frac{m_H^2}{m^2}\Big)\, L_\rho\ ,\\
&l^{(1)}_{5\rho} = l^{(1)}_{6\rho} = -2\, L_\rho\ , \quad l^{(1)}_{7\rho} = l^{(1)}_{8\rho} = l^{(1)}_{9\rho} = \Big(2 - 6\frac{m^2}{m_H^2} - 5\frac{m_H^2}{m^2}\Big)\, L_\rho
\end{aligned} \tag{4.24}$$

and $l^{(1)}_{11\rho} = 0$, where $L_\rho := \frac{1}{8\pi^2}\log\rho$, the number l_1 is obtained on computing $l^{(1)}_{1\rho} =: l_1\, L_\rho$ by method (A), and $\beta_1 := i8\pi^2\,\alpha_3$, $\beta_2 := i8\pi^2\,\alpha_4$, $\beta_3 := i8\pi^2\,\alpha_6 = -i8\pi^2\,\alpha_7 \in \mathbb{C}$ are parameters with arbitrary values.

The family (4.24) is by far not the general solution of the conditions (2.30)-(2.40); in particular, there is the trivial solution $z_\rho(L) = \frac{1}{\hbar}\,(L + \mathcal{O}(\hbar^2))$

(i.e. $e_\rho^{(1)} = 0 \; \forall e$), which is obtained by renormalizing all 1-loop-diagrams by method (B).

To 1-loop order one can even find a non-trivial solution of the clearly stronger property of BRST-invariance of $(L_0 + z_\rho(L))$; but this requires a very specific combination of the methods (A) and (B) for the various 1-loop diagrams and suitable finite renormalizations. Hence, in general, $s(L_0+z_\rho(L))$ is *not* $\simeq 0$, and also $s_0 L_0$ is *not* $\simeq 0$; in particular these two statements hold for the family (4.24) – see [11].

4.5. Frequently used renormalization schemes

In conventional momentum space renormalization a frequently used renormalization scheme is dimensional regularization with minimal subtraction, which preserves BRST-invariance generically. Applied to the 1-loop diagrams of our initial model, this property implies that the resulting time-ordered products fulfill PGI.[4] Dimensional regularization needs a mass scale $M > 0$; which remains in the formulas when removing the regularization by using minimal subtraction, and plays the role of the renormalization mass scale. Usually M is chosen according to method (A); and the minimal subtraction prescription forbids to perform any finite renormalization. Therefore, using this prescription, the Higgs mechanism is not applicable at an arbitrary scale, because the second crucial necessary condition (4.23) is violated. Relaxing this prescription by admitting the finite PGI-preserving renormalizations (4.13)-(4.21), the violation of (4.23) cannot be removed.

Another state independent renormalization scheme is the central solution of Epstein and Glaser [12]. (For 1-loop diagrams this scheme corresponds to BPHZ-subtraction at $p = 0$.) Since the subtraction point $p = 0$ is scaling invariant, the central solution maintains homogeneous scaling (w.r.t. $(x, \mathbf{m}) \to (\rho x, \mathbf{m}/\rho)$; cf. [10, Sec. 2.3]); hence, the pertinent RG-flow is trivial.

In the conventional literature one meets also state dependent renormalization conditions: e.g. in the adiabatic limit the vacuum expectation values of certain time-ordered products must agree with the "experimentally" known values for the masses of stable particles in the vacuum, and analogous conditions for parameters of certain vacuum correlation functions. Since "experimental" results are not subject to our scaling transformation, a lot of diagrams are renormalized by method (A), if we use such a scheme. To 1-loop level, the validity of the Higgs mechanism at all scales amounts then mainly to the question: is it nevertheless possible to fulfill the second crucial necessary condition (4.23), which requires to renormalize certain diagrams by method (B)?

[4]We are not aware of a proof of this statement, but it is very plausible. A corresponding statement for higher loop diagrams involves a partial adiabatic limit, because such diagrams contain inner vertices, which are integrated out with $g(x) = 1$ in conventional momentum space renormalization – but PGI is formulated *before* the adiabatic limit $g \to 1$ is taken.

5. Summary and conclusions

In the Epstein-Glaser framework the obvious way to define the RG-flow is to use the Main Theorem in the adiabatic limit [18, 14, 4, 3]: the effect of a scaling transformation (scaling with $\rho > 0$) can equivalently be expressed by a renormalization of the interaction: $L \mapsto z_\rho(L)$. The so defined RG-flow $\rho \mapsto z_\rho(L)$ depends on the choice of the renormalization mass scale(s) $M > 0$ for the various UV-divergent Feynman diagrams: if M is subject to our scaling transformation (method (B)) – e.g. the mass of one of the basic fields – the pertinent diagram does not contribute to the RG-flow. In contrast, if M is a fixed mass scale (method (A)), the corresponding diagram yields a unique (i.e. M-independent), non-vanishing contribution.

Performing the renormalization of the wave functions, masses, gauge-fixing parameter and coupling parameters, we obtain a description of the scaled model $L_0+z_\rho(L)$ (L_0 denotes the free Lagrangian) by a new Lagrangian $L_0^\rho + L^\rho$, which has essentially the same form as the original one, $L_0 + L$; but the basic fields and the parameters are ρ-dependent. The title of this paper can be reformulated as follows: is the new Lagrangian $L_0^\rho + L^\rho$ derivable by the Higgs mechanism for all $\rho > 0$?

We have investigated this question for the $U(1)$-Higgs model to 1-loop order. We only admit renormalizations of the initial model which fulfill a suitable form of BRST-invariance of the time-ordered products – we work with PGI (3.13). The answer depends not only on the choice of the renormalization method ((A) or (B)) for the various 1-loop Feynman diagrams; the RG-flow can also be modified by finite, PGI-preserving renormalizations (4.12) of the initial model. Using this non-uniqueness, we have shown that one can achieve that the Higgs mechanism is possible at all scales; one can even fulfill the much stronger condition of BRST-invariance of $L_0 + z_\rho(L)$. But this requires a quite (Higgs mechanism) or very (BRST-invariance) specific prescription for the choice of the renormalization method ((A) or (B)) for the various Feynman diagrams, and for the finite renormalizations. If one uses always method (A) – minimal subtraction is of this kind – the geometrical interpretation is violated by terms $\sim A\partial B$; weakening this prescription by admitting finite PGI-preserving renormalizations, these $A\partial B$-terms cannot be removed.

If one accepts the Higgs mechanism as a fundamental principle explaining the origin of mass at all scales (although it is not understood in a pure QFT framework), our results exclude quite a lot of renormalization schemes, in particular minimal subtraction.

On the other hand we give a model-independent proof, which uses rather weak assumptions, that the RG-flow is compatible with a weak form of BRST-invariance of the time-ordered products, namely PC (Theorem 3.1). However, in [11] it is shown that the somewhat stronger property of PGI gets lost under the RG-flow in general, and in particular if one uses a renormalization prescription corresponding to minimal subtraction.

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