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Local operators in the Sine-Gordon model: $\partial_\mu\phi\,\partial_\nu\phi$ and the stress tensor

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Abstract: We consider the simplest non-trivial local composite operators in the massless Sine-Gordon model, which are $\partial_\mu\phi\,\partial_\nu\phi$ and the stress tensor $T_{\mu\nu}$. We show that even in the finite regime $\beta^2 < 4\pi$ of the theory, these operators need additional renormalisation (beyond the free-field normal-ordering) at each order in perturbation theory. We further prove convergence of the renormalised perturbative series for their expectation values, both in the Euclidean signature and in Minkowski space-time, and for the latter in an arbitrary Hadamard state. Lastly, we show that one must add a quantum correction (proportional to $\hbar$) to the renormalised stress tensor to obtain a conserved quantity.

1. Introduction

The Sine–Gordon model is a well-studied example of a two-dimensional interacting quantum field theory. Its classical Euclidean action is given by

$$S[\phi] = \int \left[\frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \frac{1}{2}m_0^2\phi^2 - g(V_\beta + V_{-\beta}) \right] \mathrm{d}^2x \,, \tag{1}$$

where $V_{\pm\beta} \equiv \mathrm{e}^{\pm\mathrm{i}\beta\phi}$ are the vertex operators, $m_0 \geq 0$ is a mass parameter, $\beta > 0$ is the coupling constant and g is the interaction cutoff. While a priori, one takes g as a function of compact support or rapid decay (Schwartz function), ultimately one is interested in the adiabatic or infinite-volume limit $g \to$ const. The quantisation of the Sine–Gordon model, in various ranges of the parameters and using diverse approaches, has been treated by many authors, with the earliest results in the framework of Euclidean Constructive Quantum Field Theory. It turns out that for $\beta^2 < 4\pi$ (the finite regime), after Wick ordering of the vertex operators a convergent perturbation theory in g is obtained. In this regime and for sufficiently large $|m_0/g|$, Fröhlich and Seiler [1–3] proved convergence of the perturbation expansion of the Euclidean correlation functions of the field ϕ and the vertex operators $V_{\pm\beta}$ in infinite volume. In their proof, the mass term was essential to regulate the infrared problems that appear for the massless free scalar field. They also showed the existence of single-particle states and non-trivial scattering.

Still in the finite regime $\beta^2 < 4\pi$, the existence of the massless limit for the infinite-volume Euclidean correlation functions of vertex operators $V_{\pm\beta}$ and the derivative of the interacting field $\partial_\mu \phi$ was shown by Fröhlich and Park [4,5], who also proved that the Osterwalder–Schrader axioms are satisfied. For $4\pi \leq \beta^2 < 8\pi$ (the super-renormalisable regime), Wick ordering is not sufficient anymore, and a new divergent term that needs to be renormalised appears in perturbation theory each time β^2 crosses a threshold $n/(n+1)8\pi$. In this regime, the ultraviolet stability of the Sine–Gordon model (i.e., the convergence of the renormalised partition function in finite volume together with exponential bounds) has been shown by Benfatto, Gallavotti, Nicolò, Renn and Steinmann [6–8] using cluster expansions and Dimock and Hurd [9, 10] and Renn and Steinmann [11] using renormalisation group techniques, for both the massive and the massless case. For $\beta^2 < 16/3\pi$ (the second threshold), Dimock [12] has also shown the existence of correlation functions of the vertex operators in finite volume. Using Hamilton–Jacobi-like (or Wilson–Polchinski) flow equations, Brydges and Kennedy [13] showed convergence of the partition function in the massive case and infinite volume for $\beta^2 < 16/3\pi$, and Bauerschmidt and Bodineau [14] extended their results to $\beta^2 < 6\pi$.

Finally, for $\beta^2 = 8\pi$ the Sine–Gordon model becomes strictly renormalisable. Here, only perturbative renormalisability has been proven by Nicolò and Perfetti [15], while the non-perturbative existence of the model is unknown.

In the full range $0 \leq \beta^2 \leq 8\pi$, the Sine-Gordon model has been conjectured to be equivalent to the Thirring model, entailing a boson-fermion equivalence (Coleman's equivalence [16], see also [17,18]). Under this equivalence, correlation functions of vertex operators and derivatives of ϕ in the Sine–Gordon model are equal to correlation functions of fermion bilinears and currents in the Thirring model if the parameters of both models are suitably identified; in particular, the Sine–Gordon coupling g is identified with the mass of the fermion while β is related to the current-current coupling λ in the Thirring model. This equivalence has been proven for $\beta^2 < 4\pi$ by Fröhlich and Seiler [3] in the massive case, by Benfatto, Falco and Mastropietro [19] in the massless case and in finite volume, and by Dimock [12] for $\beta^2 = 4\pi$ (where the Thirring model becomes free) also in finite volume. Only recently, Bauerschmidt and Webb [20] achieved a proof of this correspondence for the massless case in infinite volume, also for $\beta^2 = 4\pi$.

On the other hand, the classical massless Sine–Gordon model is integrable and one expects that integrability survives quantisation, such that the S-matrix factorises into two-particle scattering functions. Their form for the Sine–Gordon model has been conjectured by Zamolodchikov and Zamolodchikov [21], but the integrable structure and the factorisation of the S-Matrix are not visible in the previous constructions. The conjectured S-matrix of the massless Sine–Gordon model has been studied in the form factor programme by Babujian, Karowski and collaborators [22, 23]. In this approach one computes Wightman n-point functions of interacting pointlike local fields in terms of certain matrix components ("form factors") of these. However, one has to deal with infinite expansions whose convergence is difficult to control, and therefore the existence of the fields themselves is currently out of reach in this approach.

A rigorous construction of the massless Sine–Gordon model has been achieved directly in Minkowski spacetime in the framework of perturbative Algebraic Quantum Field Theory by Bahns and Rejzner [24]. In the finite regime, they proved that the perturbation series for the S-matrix with fixed interaction cutoff, as well as the derivative of the interacting field $\partial_\mu \phi$ and the vertex operators $V_{\pm\beta}$, which are given as formal power series both in the coupling g and in $\hbar$, converge. However, the expected factorization of the S-matrix has not been shown, which is probably only visible in the adiabatic (infinite volume) limit. In a later paper together with Fredenhagen [25], the authors also constructed a family of unitary operators (relative S-matrices) which generate the local algebras of observables (vertex operators and derivative of ϕ) of the model, and discussed the equivalence with the massive Thirring model.

In the general framework of Algebraic Quantum Field Theory, an alternative new approach to the construction of integrable quantum field theories has been carried out by Lechner starting from an idea of Schroer [26]. In this approach, a model is characterized in terms of its C^*-algebras of local observables obeying certain consistency conditions (Haag-Kastler axioms). The factorized S-matrix is an input to the construction of the theory. This approach uses as its starting point observables localized in wedge regions (wedge commutativity) and shows existence of strictly local observables in a second step, using abstract methods based on the theory of von Neumann algebras. This leads to a fully rigorous construction of the theory for a large class of scalar S-matrices [27].While this class does not include the massless Sine–Gordon model, there are recent steps towards the Sine–Gordon model by Cadamuro and Tanimoto [28]. A characterisation of local observables on the level of expansion coefficients into an infinite series of interacting creation and annihilation operators has been carried out for scalar S-matrix models by Bostelmann and Cadamuro [29]; for the massive Ising model, this leads to a rigorous construction of local observables [30]. However, outside these special cases, showing convergence of the series remains difficult.

The Sine–Gordon model has also been studied in the framework of stochastic quantisation, where one introduces a stochastic partial differential equation depending on an auxiliary ("stochastic") time τ. Computing equal-time stochastic expectation values of the solutions to this stochastic PDE, the Euclidean correlation functions in the quantum theory arise in the limit $\tau \to \infty$. Using Hairer's framework of regularity structures to solve the stochastic PDE for the Sine–Gordon model, Chandra, Hairer and Shen [31, 32] have shown the short-time existence of solutions in the finite and super-renormalisable regime $0 \leq \beta^2 < 8\pi$, but the existence of the infinite-time limit is unproven so far. The measure of the massive Sine–Gordon model in finite volume and for $\beta^2 < 4\pi$ has also been constructed using stochastic control techniques by Oh, Robert, Sosoe and Wang [33] and by Barashkov [34].

In this paper, we consider the simplest local composite operators of the massless ($m_0 = 0$) Sine–Gordon model (beyond vertex operators and the derivative of the field), which are $\mathcal{O}_{\mu\nu} \equiv \partial_\mu \phi \, \partial_\nu \phi$ and the stress tensor $T_{\mu\nu} \equiv \mathcal{O}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\mathcal{O}_\rho{}^\rho + g\eta_{\mu\nu}(V_\beta + V_{-\beta})$ (with $\eta_{\mu\nu}$ replaced by $\delta_{\mu\nu}$ in the Euclidean case). We consider both Euclidean and Minkowski signature and work in the finite regime ($\beta^2 < 4\pi$) of the massless theory. To that end, we use the well-known Gell-Mann–Low formula for the perturbation series of interacting fields, and show convergence of the renormalized perturbation series with an adiabatic interaction cutoff g, after removal of the initial IR and UV cutoffs. While we do not attempt to remove the adiabatic cutoff, we show convergence for arbitrary Schwartz functions g, which in the Minkowski case represents a technical improvement over [24, Thm. 6], whose results only hold if the support of g is small enough.

Our results are as follows: We start with the case of Euclidean signature. In view of the Gell-Mann–Low formula, we first show that the renormalised expectation values of $\mathcal{O}_{\mu\nu}$ and $T_{\mu\nu}$ are well-defined (in the sense of distributions) after removal of the IR and UV cutoffs:

Theorem 1 (Renormalisation in Euclidean signature). *Consider the massless Euclidean Sine-Gordon model in the finite regime $\beta^2 < 4\pi$ and with the free-field covariance $C^{\Lambda,\epsilon}$ with IR cutoff Λ and UV cutoff ϵ. There exists a choice of local counterterms (diverging logarithmically with the UV cutoff ϵ) such that the renormalised expectation values*

$$\left\langle \mathcal{N}_\mu[\mathcal{O}_{\mu\nu}(z)] \prod_{j=1}^{n} \mathcal{N}_\mu\left[V_{\sigma_j\beta}(x_j)\right] \right\rangle_{0,ren}^{\Lambda,\epsilon} \tag{2}$$

in the free theory, with $\mathcal{N}_\mu$ denoting normal ordering with respect to the covariance $C^{\mu,\epsilon}$ and $\sigma_j = \pm 1$, are well-defined distributions in the physical limit $\Lambda, \epsilon \to 0$. For the stress tensor, the physical limit

$$\lim_{\Lambda,\epsilon\to 0} \left\langle \mathcal{N}_\mu[T_{\mu\nu}(z)] \prod_{j=1}^{n} \mathcal{N}_\mu\left[V_{\sigma_j\beta}(x_j)\right] \right\rangle_{0,ren}^{\Lambda,\epsilon} \tag{3}$$

exists without counterterms. The expectation values involving $\mathcal{O}_{\mu\nu}$ vanish in the physical limit unless the neutrality condition $\sum_{j=1}^{n} \sigma_j = 0$ is fulfilled, while the ones involving the stress tensor vanish unless $\sum_{j=1}^{n} \sigma_j \in \{-1, 0, 1\}$.

We then show convergence of the renormalised Gell-Mann–Low perturbation series with an adiabatic cutoff in the physical limit, i.e., for vanishing IR and UV cutoffs:

Theorem 2 (Convergence of the renormalised perturbation series). *Under the same assumptions as in Theorem 1 and with a non-negative adiabatic cutoff function $0 \leq g \in \mathcal{S}(\mathbb{R}^2)$, the perturbative series for the (normalised, interacting) Gell-Mann–Low expectation value of $\mathcal{O}_{\mu\nu}$*

$$\begin{aligned}&\langle \mathcal{N}_\mu[\mathcal{O}_{\mu\nu}(f)] \rangle_{int,ren} \equiv \\ &\frac{\sum_{n=0}^{\infty} \frac{1}{n!} \int \cdots \int \sum_{\sigma_i=\pm 1} \left\langle \mathcal{N}_\mu[\mathcal{O}_{\mu\nu}(f)] \prod_{j=1}^{n} \mathcal{N}_\mu\left[V_{\sigma_j\beta}(x_j)\right] \right\rangle_{0,ren}^{0,0} \prod_{i=1}^{n} g(x_i)\, \mathrm{d}^2 x_i}{\sum_{n=0}^{\infty} \frac{1}{n!} \int \cdots \int \sum_{\sigma_i=\pm 1} \left\langle \prod_{j=1}^{n} \mathcal{N}_\mu\left[V_{\sigma_j\beta}(x_j)\right] \right\rangle_{0,ren}^{0,0} \prod_{i=1}^{n} g(x_i)\, \mathrm{d}^2 x_i}\end{aligned} \tag{4}$$

is convergent, where the operator $\mathcal{O}_{\mu\nu}$ is smeared with a test function $f \in \mathcal{S}(\mathbb{R}^2)$ and the physical limit $\Lambda, \epsilon \to 0$ is taken termwise. There exists a constant $K > 0$

(depending on g and β) and a constant $C > 0$ (depending on f) such that the series is bounded by

$$C\sum_{n=0}^{\infty} n^2 K^n \begin{Bmatrix} (n!)^{-1} & \beta^2 < 2\pi \\ (n!)^{\frac{\beta^2}{2\pi}-2} & 2\pi \leq \beta^2 < 4\pi \end{Bmatrix} < \infty\,. \tag{5}$$

The same holds for the smeared stress tensor $T_{\mu\nu}(f)$, with the constant C also depending on g.

Lastly, we show that a modified stress tensor, obtained by a rescaling of the coupling constant, fulfills the continuity equation in the quantum theory:

Theorem 3 (Conservation of the stress tensor). *Under the same assumptions as in Theorem 2, a modified stress tensor $\hat{T}_{\mu\nu}$ is conserved in the quantum theory: we have*

$$\left\langle \mathcal{N}_\mu \left[\hat{T}_{\mu\nu}(\partial^\mu f)\right]\right\rangle_{int,ren} = 0 \tag{6}$$

for all $f \in \mathcal{S}(\mathbb{R}^2)$ such that g is constant on the support of f. The required modification is a rescaling of the coupling g:

$$\hat{T}_{\mu\nu} = \mathcal{O}_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}\mathcal{O}_\rho{}^\rho + g\left(1 - \frac{\beta^2}{8\pi}\right)\delta_{\mu\nu}(V_\beta + V_{-\beta})\,. \tag{7}$$

Similar results apply to the case of Minkowski signature. Namely, we show that the renormalised expectation values of time-ordered products involving $\mathcal{O}_{\mu\nu}$ and $T_{\mu\nu}$ in any quasi-free Hadamard state, regularized with IR and UV cutoffs, are well-defined in the sense of distributions when the cutoffs are removed. We prove:

Theorem 4 (Renormalisation in Minkowski space-time). *Consider the massless Lorentzian Sine-Gordon model in the finite regime $\beta^2 < 4\pi$ and a quasi-free state $\omega^{\Lambda,\epsilon}$ in the vacuum sector whose two-point function has an IR cutoff Λ and UV cutoff ϵ. There exists a choice of local counterterms (diverging logarithmically with the UV cutoff ϵ) such that the renormalised expectation values of time-ordered products*

$$\omega^{\Lambda,\epsilon}\left(\mathcal{T}\left[\mathcal{O}_{\mu\nu}(z) \otimes \bigotimes_{j=1}^{n} V_{\sigma_j\beta}(x_j)\right]\right) \tag{8}$$

with $\sigma_j = \pm 1$ in the free theory are well-defined distributions in the physical limit $\Lambda, \epsilon \to 0$. For the stress tensor, the physical limit

$$\lim_{\Lambda,\epsilon\to 0} \omega^{\Lambda,\epsilon}\left(\mathcal{T}\left[T_{\mu\nu}(z) \otimes \bigotimes_{j=1}^{n} V_{\sigma_j\beta}(x_j)\right]\right) \tag{9}$$

exists without counterterms. The expectation values involving $\mathcal{O}_{\mu\nu}$ vanish in the physical limit unless the neutrality condition $\sum_{j=1}^{n}\sigma_j = 0$ is fulfilled, while the ones involving the stress tensor vanish unless $\sum_{j=1}^{n}\sigma_j \in \{-1, 0, 1\}$.

In analogy to the Euclidean case, we then show convergence of the renormalised perturbation series given by the Gell-Man–Low formula for $\mathcal{O}_{\mu\nu}$ and $T_{\mu\nu}$ with interaction cutoff, and under an additional assumption on the state-dependent part of the two-point function of the quasi-free state.

Theorem 5 (Convergence of the renormalised perturbation series). *We make the same assumptions as in Theorem 4, and in addition require that the state-dependent part W of the two-point function of the state ω satisfies:*

1. *$W(x,y)$ and its first and second derivatives grow at most polynomially,*
2. *$\sum_{i,j=1}^{n} [W(x_i,x_j) - W(y_i,x_j) - W(x_i,y_j) + W(y_i,y_j)] \geq 0$ for any configuration of points x_i and y_i and any $n \in \mathbb{N}$.*

Then for any adiabatic cutoff function $g \in \mathcal{S}(\mathbb{R}^2)$, the perturbative series for the (normalised, interacting) Gell-Mann–Low expectation value

$$\omega_{int}(\mathcal{O}_{\mu\nu}(f)) \equiv \frac{\sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} \int \cdots \int \sum_{\sigma_i=\pm 1} \omega^{0,0}\Big(\mathcal{T}\Big[\mathcal{O}_{\mu\nu}(f) \otimes \bigotimes_{j=1}^{n} V_{\sigma_j \beta}(x_j)\Big]\Big) \prod_{i=1}^{n} g(x_i)\, \mathrm{d}^2 x_i}{\sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} \int \cdots \int \sum_{\sigma_i=\pm 1} \omega^{0,0}\Big(\mathcal{T}\Big[\bigotimes_{j=1}^{n} V_{\sigma_j \beta}(x_j)\Big]\Big) \prod_{i=1}^{n} g(x_i)\, \mathrm{d}^2 x_i} \tag{10}$$

is convergent, where the operator $\mathcal{O}_{\mu\nu}$ is smeared with a test function $f \in \mathcal{S}(\mathbb{R}^2)$ and the physical limit $\Lambda, \epsilon \to 0$ is taken termwise. There exists a constant $K > 0$ (depending on g, β and W) and a constant $C > 0$ (depending on f, β and W) such that the series is bounded by

$$C \sum_{n=0}^{\infty} n^2 K^n (n!)^{\frac{\beta^2}{4\pi} - 1} < \infty \tag{11}$$

if $\|g\|_\infty$ is small enough (depending on g and β). The same holds for the smeared stress tensor $T_{\mu\nu}(f)$, with the constant C also depending on g.

Remark. While the first condition on the state-dependent part W of the two-point function is very reasonable, the second one might seem strange at first sight. It is however necessary for convergence of the perturbative series, and one can easily construct a wide range of Hadamard states that fulfill it. For example, any state for which W can be written as

$$W(x,y) = \int \mathrm{e}^{\mathrm{i}p(x-y)}\, \mathrm{d}\mu_W(p) \tag{12}$$

with a positive measure $\mathrm{d}\mu_W(p)$ fulfills the condition, since then

$$\begin{aligned} &\sum_{i,j=1}^{n} [W(x_i,x_j) - W(y_i,x_j) - W(x_i,y_j) + W(y_i,y_j)] \\ &\quad = \int \left| \sum_{i=1}^{n} \left(\mathrm{e}^{\mathrm{i}p x_i} - \mathrm{e}^{\mathrm{i}p y_i}\right) \right|^2 \mathrm{d}\mu_W(p) \geq 0\,. \end{aligned} \tag{13}$$

Examples of such states are states that are thermal in a wide range of energies between E_0 and E_1, where

$$\begin{aligned}\mathrm{d}\mu_W(p) &= \Theta\big(\left|p^0\right| \in [E_0, E_1]\big) \int \frac{\mathrm{e}^{-\beta\left|p^1\right|}}{|p^1|\big(1-\mathrm{e}^{-\beta|p^1|}\big)} \cos[p^1(t-t')]\mathrm{e}^{\mathrm{i}p^0(t-t')}\,\mathrm{d}t \frac{\mathrm{d}^2 p}{(2\pi)^2} \\ &= \Theta\big(\left|p^0\right| \in [E_0, E_1]\big) \frac{\pi \mathrm{e}^{-\beta\left|p^0\right|}}{|p^0|\big(1-\mathrm{e}^{-\beta|p^0|}\big)} [\delta(p^1+p^0)+\delta(p^1-p^0)] \frac{\mathrm{d}^2 p}{(2\pi)^2}\,,\end{aligned} \tag{14}$$

and the last line is clearly a positive measure. Here the restriction on the energy range is necessary, since we consider a massless theory where thermal states do not exist for all energies, which is seen from the singularity of the prefactor for small $\left|p^0\right|$. On the other hand, for the massive Sine–Gordon model true thermal states have recently been constructed in [35].

Finally, we show that analogous to the Euclidean case we need to modify the stress tensor by a rescaling of the coupling constant, such that it fulfills the continuity equation also in the Minkowski signature. This modification agrees with the one proposed in the form factor programme [23], and thus proves its correctness.

Theorem 6 (Conservation of the stress tensor). *Under the same assumptions as in Theorem 5, there exists a redefinition of time-ordered products* $\mathcal{T}[\mathcal{O}_{\mu\nu}(x) \otimes V_\alpha(y)] \to \mathcal{T}[\mathcal{O}_{\mu\nu}(x) \otimes V_\alpha(y)] + \delta\mathcal{T}[\mathcal{O}_{\mu\nu}(x) \otimes V_\alpha(y)]$ *with* $\delta\mathcal{T}$ *a local term proportional to* $\delta^2(x-y)\mathcal{T}[V_\alpha(y)]$*, such that the modified stress tensor*

$$\hat{T}_{\mu\nu} \equiv \mathcal{O}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\mathcal{O}_\rho{}^\rho + g\left(1-\frac{\beta^2}{8\pi}\right)\eta_{\mu\nu}(V_\beta + V_{-\beta}) \tag{15}$$

is conserved in the quantum theory:

$$\omega_{int}\Big(\hat{T}_{\mu\nu}(\partial^\mu f)\Big) = 0 \tag{16}$$

for all $f \in \mathcal{S}(\mathbb{R}^2)$ *such that* g *is constant on the support of* f*.*

The remaining sections are dedicated to the proof of these theorems.

Our conventions are as follows: We set $\hbar = c = 1$. Note that if we would keep $\hbar$ explicit, it would only arise in the combination $\hbar\beta^2$, so that for example the modifications of the stress tensor $\hat{T}_{\mu\nu}$ are really quantum modifications. Interestingly, they are one-loop exact, i.e., no terms of order $\hbar^2$ or higher arise.

An obvious extension of our work would be the generalisation of the results to the higher conserved currents of the Sine–Gordon model which exist classically as a consequence of integrability, as well as in perturbation theory [36–39]. Their classical expression is obtained from an explicit recurrence relation, and in the form factor programme it has been proposed that — in contrast to the stress tensor — they do not receive quantum corrections [23]. Another important step that is still unclear is the adiabatic (infinite-volume) limit $g \to$ const, which already for correlation functions of the vertex operators is a difficult task. To show the existence of this limit, one probably has to first show the non-perturbative generation of a finite mass (Debye screening), which is known to arise at least for small $|\beta g|$ [40, 41] and in the case $\beta^2 = 4\pi$ [20]. Also the extension of our results to the super-renormalisable range $4\pi \leq \beta^2 < 8\pi$ is a worthwhile task to achieve in the future. Lastly, while the equivalence with the Thirring model has been proven for correlation functions of the vertex operators and derivatives of the field, their equality for correlation functions involving the stress tensor or higher-order conserved currents (which exist in the Thirring model [38, 39]) is so far unknown, and needs to be studied.

2. Euclidean case

In the construction of the theory in Euclidean signature, we follow the well-established treatment of Euclidean quantum field theories via functional integrals. An introduction can be found in [42], from which we take formulas and results without specifying their source explicitly.

2.1. Preliminaries. We consider a centred Gaussian measure $\mathrm{d}\mu^{\Lambda,\epsilon}(\phi)$ with covariance $C^{\Lambda,\epsilon}(x, y)$ depending on an IR cutoff Λ and a UV cutoff Λ_0. Centred and Gaussian means that

$$\int \phi(x_1) \cdots \phi(x_n) \, \mathrm{d}\mu^{\Lambda,\epsilon}(\phi) = \sum_\pi \prod_{(i,j)\in\pi} C^{\Lambda,\epsilon}(x_i, x_j) \,, \tag{17}$$

where the sum runs over all partitions π of the set $\{1, \ldots, n\}$ into unordered pairs (i, j). Therefore, the right-hand side vanishes if n is odd. In general, we have the characteristic function

$$\int \mathrm{e}^{\mathrm{i}(J,\phi)} \, \mathrm{d}\mu^{\Lambda,\epsilon}(\phi) = \mathrm{e}^{-\frac{1}{2}(J, C^{\Lambda,\epsilon} * J)} \tag{18}$$

where we introduced the scalar product

$$(f, g) \equiv \int f(x)g(x) \, \mathrm{d}^2x \tag{19}$$

and the convolution

$$(f * g)(x) \equiv \int f(x, y)g(y) \, \mathrm{d}^2y \,, \tag{20}$$

from which the above follows by functional differentiation with respect to J. For finite cutoffs, the covariance is assumed to be a smooth function, and hence the Gaussian measure is supported on smooth functions $\phi \in \mathcal{S}(\mathbb{R}^2)$. In general, and in particular as the cutoffs are removed, the measure is supported on distributions. However, the combinatorics are unaffected by the support properties of the measure. Obviously, as the cutoffs are removed in the physical limit $\Lambda, \epsilon \to 0$, the covariance must turn into the free propagator. Regulated expectation values in the free theory are defined by the Gaussian integral (17)

$$\langle A_1[\phi] \cdots A_n[\phi] \rangle_0^{\Lambda,\epsilon} \equiv \int A_1[\phi] \cdots A_n[\phi] \, \mathrm{d}\mu^{\Lambda,\epsilon}(\phi) \,, \tag{21}$$

where the A_i are local functionals of the field, and the interacting expectation values are computed from the Gell-Man–Low formula

$$\langle A_1[\phi]\cdots A_n[\phi]\rangle_{\text{int}}^{\Lambda,\epsilon} \equiv \frac{\left\langle A_1[\phi]\cdots A_n[\phi]\mathrm{e}^{-S_{\text{int}}[\phi]}\right\rangle_0^{\Lambda,\epsilon}}{\left\langle \mathrm{e}^{-S_{\text{int}}[\phi]}\right\rangle_0^{\Lambda,\epsilon}}\,, \tag{22}$$

where S_{int} is the interaction which is assumed to be bounded from below.

It is well known that the massless scalar field in two dimensions is not well defined because of infrared problems [43, 44]. In our case, this manifests as a logarithmic divergence of the regulated covariance

$$C^{\Lambda,\epsilon}(x,y) \equiv -\frac{1}{4\pi}\ln\left[\Lambda^2\left[(x-y)^2+\epsilon^2\right]\right] \tag{23}$$

as the IR cutoff Λ is removed. To verify that (23) is a suitable regularisation, we note that the UV cutoff ϵ obviously regulates the UV singular behaviour as $x \to y$, while the IR cutoff Λ arises from the small-mass limit of the massive propagator:

$$\begin{aligned}\int \frac{\mathrm{e}^{\mathrm{i}p(x-y)}}{p^2+m^2}\frac{\mathrm{d}^2p}{(2\pi)^2} &= \frac{1}{2\pi}\int_0^\infty \frac{q}{q^2+m^2}\,\mathrm{J}_0(q|x-y|)\,\mathrm{d}q \\ &= \frac{1}{2\pi}\,\mathrm{K}_0(m|x-y|) = -\frac{1}{4\pi}\ln\left[\frac{m^2\mathrm{e}^{2\gamma}}{4}(x-y)^2\right] + \mathcal{O}(m)\,.\end{aligned} \tag{24}$$

We will see later on that this divergence is responsible for the super-selection sectors of the theory, and for the vacuum sector which we consider results in a neutrality condition [45].

In the range $\beta^2 < 4\pi$, one only needs to normal-order the interaction to obtain finite correlation functions of the basic field ϕ. Normal-ordering $\mathcal{N}$ is a linear operation that can be defined with respect to any given covariance, but we only need it with respect to the covariance $C^{\mu,\epsilon}$ (23) for a fixed IR cutoff μ. We have the explicit formula for exponentials

$$\mathcal{N}_\mu\left[\mathrm{e}^{\mathrm{i}(J,\phi)}\right] = \mathrm{e}^{\frac{1}{2}(J,C^{\mu,\epsilon}*J)}\mathrm{e}^{\mathrm{i}(J,\phi)}\,, \tag{25}$$

from which normal-ordering of monomials can be obtained by functional differentiation with respect to J. Moreover, for $J(y) = \pm\beta\delta^2(y-x)$ we obtain the normal-ordering of the interaction of the Sine-Gordon theory:

$$\mathcal{N}_\mu[2\cos(\beta\phi(x))] = \mathcal{N}_\mu\left[\mathrm{e}^{\mathrm{i}\beta\phi(x)}\right] + \mathcal{N}_\mu\left[\mathrm{e}^{-\mathrm{i}\beta\phi(x)}\right]\,, \tag{26a}$$

$$\mathcal{N}_\mu\left[\mathrm{e}^{\mathrm{i}\alpha\phi(x)}\right] = \mathrm{e}^{\frac{1}{2}\alpha^2C^{\mu,\epsilon}(x,x)}\mathrm{e}^{\mathrm{i}\alpha\phi(x)} = (\mu\epsilon)^{-\frac{\alpha^2}{4\pi}}\mathrm{e}^{\mathrm{i}\alpha\phi(x)}\,. \tag{26b}$$

Normal-ordering has the property that

$$\langle\mathcal{N}_\Lambda[(J_1,\phi)\cdots(J_n,\phi)]\rangle_0^{\Lambda,\epsilon} = \int \mathcal{N}_\Lambda[(J_1,\phi)\cdots(J_n,\phi)]\,\mathrm{d}\mu^{\Lambda,\epsilon}(\phi) = \delta_{n,0}\,, \tag{27}$$

which makes evaluation easy, and is the reason for the name "normal-ordering". For a normal ordering with respect to a different covariance, one first has to change the normal-ordering according to

$$\begin{aligned} &\mathcal{N}_\nu[(J_1,\phi)\cdots(J_n,\phi)] \\ &\quad= \exp\left[-\frac{1}{2}\left(\frac{\delta}{\delta\phi},(C^{\nu,\epsilon}-C^{\mu,\epsilon})*\frac{\delta}{\delta\phi}\right)\right]\mathcal{N}_\mu[(J_1,\phi)\cdots(J_n,\phi)]\,, \end{aligned} \tag{28}$$

which is proven as in [42, Thm. 2.4]. We note that for unsmeared exponentials and the covariance (23), there is a particularly simple relation:

$$\mathcal{N}_\mu\left[\mathrm{e}^{\pm\mathrm{i}\beta\phi(x)}\right] = \left(\frac{\Lambda}{\mu}\right)^{\frac{\beta^2}{4\pi}}\mathcal{N}_\Lambda\left[\mathrm{e}^{\pm\mathrm{i}\beta\phi(x)}\right]. \tag{29}$$

In the proofs in the following sections, we also need a formula which (to our knowledge) first appeared in [20, Lemma 2.6], and which we generalise and formulate as a Lemma:

Lemma 1. *The expectation value of a product of exponentials and basic fields can be decomposed as*

$$\begin{aligned} &\left\langle \prod_{j=1}^m \mathrm{e}^{\mathrm{i}(\alpha_j,\phi)}\prod_{k=1}^n(\beta_k,\phi)\right\rangle_0^{\Lambda,\epsilon} = \left\langle \prod_{j=1}^m \mathrm{e}^{\mathrm{i}(\alpha_j,\phi)}\right\rangle_0^{\Lambda,\epsilon} \\ &\quad\times\prod_{k=1}^n\frac{\partial}{\partial\sigma_k}\exp\left[\mathrm{i}\sum_{j=1}^m\sum_{k=1}^n\sigma_k\big(\alpha_j,C^{\Lambda,\epsilon}*\beta_k\big)+\sum_{1\leq k<\ell\leq n}\sigma_k\sigma_\ell\big(\beta_k,C^{\Lambda,\epsilon}*\beta_\ell\big)\right]\Bigg|_{\sigma_i=0}. \end{aligned} \tag{30}$$

In particular, for $n=2$ we obtain

$$\begin{aligned} &\left\langle (\beta_1,\phi)(\beta_2,\phi)\prod_{j=1}^m \mathrm{e}^{\mathrm{i}(\alpha_j,\phi)}\right\rangle_0^{\Lambda,\epsilon} = \left\langle \prod_{j=1}^m \mathrm{e}^{\mathrm{i}(\alpha_j,\phi)}\right\rangle_0^{\Lambda,\epsilon} \\ &\quad\times\left[\big(\beta_1,C^{\Lambda,\epsilon}*\beta_2\big)-\sum_{j,k=1}^m\big(\alpha_j,C^{\Lambda,\epsilon}*\beta_1\big)\big(\alpha_k,C^{\Lambda,\epsilon}*\beta_2\big)\right]. \end{aligned} \tag{31}$$

Proof. We essentially transcribe the proof of [20] from Gaussian random variables to functional integrals. We first recall the formula for a shifted Gaussian measure

$$\mathrm{d}\mu^{\Lambda,\epsilon}\big(\phi+C^{\Lambda,\epsilon}*h\big) = \mathrm{e}^{-\frac{1}{2}(h,C^{\Lambda,\epsilon}*h)}\mathrm{e}^{-(\phi,h)}\,\mathrm{d}\mu^{\Lambda,\epsilon}(\phi) \tag{32}$$

with $h \in \mathcal{S}(\mathbb{R}^2)$, which is easily proven by showing that the characteristic functions (18) of both measures are the same. We then compute

$$\begin{aligned}
&\left\langle \prod_{j=1}^m \mathrm{e}^{\mathrm{i}(\alpha_j,\phi)} \mathrm{e}^{(h,\phi)} \right\rangle_0^{\Lambda,\epsilon} = \int \prod_{j=1}^m \mathrm{e}^{\mathrm{i}(\alpha_j,\phi)} \mathrm{e}^{(h,\phi)} \,\mathrm{d}\mu^{\Lambda,\epsilon}(\phi) \\
&= \int \prod_{j=1}^m \mathrm{e}^{\mathrm{i}(\alpha_j,\phi+C^{\Lambda,\epsilon}*h)} \exp\left[\left(h,\phi+C^{\Lambda,\epsilon}*h\right)\right] \mathrm{d}\mu^{\Lambda,\epsilon}\left(\phi+C^{\Lambda,\epsilon}*h\right) \\
&= \int \prod_{j=1}^m \mathrm{e}^{\mathrm{i}(\alpha_j,\phi+C^{\Lambda,\epsilon}*h)} \exp\left[\left(h,\phi+C^{\Lambda,\epsilon}*h\right) - \frac{1}{2}\left(h,C^{\Lambda,\epsilon}*h\right) - (\phi,h)\right] \mathrm{d}\mu^{\Lambda,\epsilon}(\phi) \\
&= \int \prod_{j=1}^m \mathrm{e}^{\mathrm{i}(\alpha_j,\phi+C^{\Lambda,\epsilon}*h)} \exp\left[\frac{1}{2}\left(h,C^{\Lambda,\epsilon}*h\right)\right] \mathrm{d}\mu^{\Lambda,\epsilon}(\phi) \\
&= \left\langle \prod_{j=1}^m \mathrm{e}^{\mathrm{i}(\alpha_j,\phi)} \right\rangle_0^{\Lambda,\epsilon} \exp\left[\mathrm{i}\sum_{j=1}^m (\alpha_j, C^{\Lambda,\epsilon}*h) + \frac{1}{2}\left(h,C^{\Lambda,\epsilon}*h\right)\right],
\end{aligned} \tag{33}$$

where we used the formula (32) in the third equality. We then set

$$h = \sum_{k=1}^n \sigma_k \beta_k \tag{34}$$

and use that

$$\prod_{k=1}^n (\beta_k,\phi) = \prod_{k=1}^n \frac{\partial}{\partial\sigma_k} \mathrm{e}^{(h,\phi)}\bigg|_{\sigma_i=0} \tag{35}$$

to obtain equation (30). Equation (31) follows immediately. □

2.2. Proof of theorem 1 (Renormalisation). We begin with $\mathcal{O}_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi$. Taking two functional derivatives of equation (25) with respect to J and setting J to zero, we obtain

$$\mathcal{N}_\mu[\phi(x)\phi(y)] = \phi(x)\phi(y) - C^{\mu,\epsilon}(x,y)\,, \tag{36}$$

and taking derivatives and setting $x = y = z$, it follows that

$$\mathcal{N}_\mu[\mathcal{O}_{\mu\nu}(z)] = \mathcal{O}_{\mu\nu}(z) + \lim_{x\to z} \partial_\mu\partial_\nu C^{\mu,\epsilon}(x,z) = \mathcal{O}_{\mu\nu}(z) - \frac{1}{2\pi\epsilon^2}\delta_{\mu\nu}\,, \tag{37}$$

using the explicit form of the covariance (23). The normal-ordering of the vertex operators $V_\alpha(x) = \mathrm{e}^{\mathrm{i}\alpha\phi(x)}$ is given in equation (26), such that we obtain (with $\sigma_j = \pm 1$)

$$\begin{aligned}
&\left\langle \mathcal{N}_\mu[\mathcal{O}_{\mu\nu}(z)] \prod_{j=1}^n \mathcal{N}_\mu\left[V_{\sigma_j\beta}(x_j)\right] \right\rangle_0^{\Lambda,\epsilon} \\
&= (\mu\epsilon)^{-n\frac{\beta^2}{4\pi}} \left\langle \left[\mathcal{O}_{\mu\nu}(z) - \frac{1}{2\pi\epsilon^2}\delta_{\mu\nu}\right] \prod_{j=1}^n V_{\sigma_j\beta}(x_j) \right\rangle_0^{\Lambda,\epsilon}.
\end{aligned} \tag{38}$$

Using Lemma 1, in particular equation (31) with $\beta_1(x) = \partial_\mu \delta^2(z-x)$, $\beta_2(x) = \partial_\nu \delta^2(z-x)$ and $\alpha_j(x) = \sigma_j \beta \delta^2(x-x_j)$, it follows that

$$\begin{aligned} &\left\langle \mathcal{N}_\mu[\mathcal{O}_{\mu\nu}(z)] \prod_{j=1}^n \mathcal{N}_\mu\left[V_{\sigma_j\beta}(x_j)\right] \right\rangle_0^{\Lambda,\epsilon} \\ &= -\beta^2 (\mu\epsilon)^{-n\frac{\beta^2}{4\pi}} \left\langle \prod_{j=1}^n V_{\sigma_j\beta}(x_j) \right\rangle_0^{\Lambda,\epsilon} \sum_{j,k=1}^n \sigma_j \sigma_k \partial_\mu C^{\Lambda,\epsilon}(x_j,z) \partial_\nu C^{\Lambda,\epsilon}(x_k,z)\,, \end{aligned} \tag{39}$$

where the term coming from the normal ordering (37) has canceled with the first term in equation (31). Finally, using equation (18) with $J(x) = \sum_{j=1}^n \sigma_j \beta \delta^2(x-x_j)$ we obtain

$$\begin{aligned} &\left\langle \mathcal{N}_\mu[\mathcal{O}_{\mu\nu}(z)] \prod_{j=1}^n \mathcal{N}_\mu\left[V_{\sigma_j\beta}(x_j)\right] \right\rangle_0^{\Lambda,\epsilon} \\ &= -\beta^2 (\mu\epsilon)^{-n\frac{\beta^2}{4\pi}} \exp\left[-\frac{1}{2}\beta^2 \sum_{j,k=1}^n \sigma_j \sigma_k C^{\Lambda,\epsilon}(x_j,x_k) \right] \\ &\quad\times \sum_{j,k=1}^n \sigma_j \sigma_k \partial_\mu C^{\Lambda,\epsilon}(x_j,z) \partial_\nu C^{\Lambda,\epsilon}(x_k,z) \\ &= -\frac{\beta^2}{4\pi^2} \left(\frac{\Lambda}{\mu} \right)^{\frac{\beta^2}{4\pi}\left(\sum_{j=1}^n \sigma_j\right)^2} \prod_{1\leq j<k\leq n} \left[\mu^2 \left[(x_j-x_k)^2 + \epsilon^2 \right] \right]^{\sigma_j\sigma_k\frac{\beta^2}{4\pi}} \\ &\quad\times \sum_{j,k=1}^n \sigma_j \sigma_k \frac{(x_j-z)_\mu}{(x_j-z)^2+\epsilon^2} \frac{(x_k-z)_\nu}{(x_k-z)^2+\epsilon^2} \end{aligned} \tag{40}$$

with the explicit form of the covariance (23).

Since we are in the finite regime $\beta^2 < 4\pi$, the terms $\left[\mu^2\left[(x_j-x_k)^2+\epsilon^2\right]\right]^{\sigma_j\sigma_k\frac{\beta^2}{4\pi}}$ are singular in the limit $\epsilon \to 0$ if $\sigma_j\sigma_k = -1$, but the singularity is integrable. The same holds for the terms $(x_j-z)_\mu/[(x_j-z)^2+\epsilon^2](x_k-z)_\nu/[(x_k-z)^2+\epsilon^2]$ if $j \neq k$, but for $j = k$ their scaling degree at $x_j = z$ is $2 = \dim \mathbb{R}^2$ in the limit $\epsilon \to 0$, such that we have a logarithmic singularity in this case and need to renormalise. We recall that the scaling degree of a distribution $u \in \mathcal{S}'(\mathbb{R}^k)$ at $x = 0$ is defined as

$$\operatorname{sd}(u) \equiv \inf\left\{ a \in \mathbb{R}\colon \lim_{\lambda\to 0} \lambda^a u(f_\lambda) = 0 \ \forall f \in \mathcal{S}(\mathbb{R}^k) \right\}, \tag{41}$$

where $f_\lambda(x) \equiv f(\lambda x)$; in this case, since $u_{\mu\nu}$ defined by

$$u_{\mu\nu}(f) \equiv \int \frac{x_\mu x_\nu}{(x^2+\epsilon^2)^2} f(x)\, \mathrm{d}^2 x \tag{42}$$

is not a well-defined distribution for $\epsilon = 0$, we have to restrict to test functions $f \in \mathcal{S}(\mathbb{R}^2 \setminus \{0\})$. To renormalise $u_{\mu\nu}$ and thus obtain a well-defined distribution, we compute for $\epsilon > 0$ that

$$\partial_\mu \partial_\nu \ln\big[\mu^2\big(x^2+\epsilon^2\big)\big] = \frac{2\delta_{\mu\nu}}{x^2+\epsilon^2} - \frac{4x_\mu x_\nu}{(x^2+\epsilon^2)^2} \,, \tag{43}$$

such that

$$\begin{aligned} u_{\mu\nu}(f) &= \int \left[\frac{1}{2}\frac{\delta_{\mu\nu}}{x^2+\epsilon^2} - \frac{1}{4}\partial_\mu\partial_\nu \ln\big[\mu^2\big(x^2+\epsilon^2\big)\big]\right] f(x)\,\mathrm{d}^2x \\ &= \frac{1}{2}\delta_{\mu\nu}\int \frac{1}{x^2+\epsilon^2} f(x)\,\mathrm{d}^2x - \frac{1}{4}\int \ln\big[\mu^2\big(x^2+\epsilon^2\big)\big]\partial_\mu\partial_\nu f(x)\,\mathrm{d}^2x \,. \end{aligned} \tag{44}$$

In the second term, we can take the limit $\epsilon \to 0$ since the singularity of the integrand at $x = 0$ is logarithmic and thus integrable. To extract the singular part from the first term and determine the required counterterms, we pass to Fourier space and compute

$$\begin{aligned} \int 2\pi\,\mathrm{K}_0(|p|\epsilon)\,\mathrm{e}^{\mathrm{i}px}\frac{\mathrm{d}^2p}{(2\pi)^2} &= (2\pi)^{-1}\int_0^\infty \int_0^{2\pi} \mathrm{K}_0(|p|\epsilon)\,\mathrm{e}^{\mathrm{i}|p||x|\cos\phi}|p|\,\mathrm{d}\phi\,\mathrm{d}|p| \\ &= \int_0^\infty \mathrm{K}_0(|p|\epsilon)\,\mathrm{J}_0(|p||x|)|p|\,\mathrm{d}|p| = \frac{1}{x^2+\epsilon^2} \end{aligned} \tag{45}$$

using the integrals [46, Eqs. (10.9.2) and (10.43.27)], where K is the second modified and J the ordinary Bessel function. Using the known expansion of the modified Bessel function for small argument [46, Eq. (10.31.2)]

$$\mathrm{K}_0(|p|\epsilon) = -\gamma - \ln\!\left(\frac{\epsilon|p|}{2}\right) + \mathcal{O}\big(\epsilon^2\ln\epsilon\big) \,, \tag{46}$$

we obtain

$$\begin{aligned} \frac{1}{x^2+\epsilon^2} &= 2\pi\int [\mathrm{K}_0(|p|\epsilon) + \ln(\mu\epsilon)]\,\mathrm{e}^{\mathrm{i}px}\frac{\mathrm{d}^2p}{(2\pi)^2} - 2\pi\ln(\mu\epsilon)\,\delta^2(x) \\ &\approx -2\pi\int \ln\!\left(\frac{|p|}{2\mu}\mathrm{e}^\gamma\right)\mathrm{e}^{\mathrm{i}px}\frac{\mathrm{d}^2p}{(2\pi)^2} - 2\pi\ln(\mu\epsilon)\,\delta^2(x) \end{aligned} \tag{47}$$

as $\epsilon \to 0$. To explicitly determine the form of the first term in coordinate space, we write

$$\begin{aligned} \int \ln\!\left(\frac{|p|}{2\mu}\mathrm{e}^\gamma\right)\mathrm{e}^{\mathrm{i}px}\frac{\mathrm{d}^2p}{(2\pi)^2} &= -\mathrm{i}\partial_\rho \int \frac{p^\rho}{p^2}\ln\!\left(\frac{|p|}{2\mu}\mathrm{e}^\gamma\right)\mathrm{e}^{\mathrm{i}px}\frac{\mathrm{d}^2p}{(2\pi)^2} \\ &= -\mathrm{i}\partial_\rho\left[x^\rho \int \frac{(px)}{p^2x^2}\ln\!\left(\frac{|p|}{2\mu}\mathrm{e}^\gamma\right)\mathrm{e}^{\mathrm{i}px}\frac{\mathrm{d}^2p}{(2\pi)^2}\right] , \end{aligned} \tag{48}$$

since by Euclidean covariance the integral must be proportional to x^ρ. Using spherical coordinates, we find

$$\begin{aligned}\int \frac{(px)}{p^2 x^2} \ln\left(\frac{|p|}{2\mu} \mathrm{e}^\gamma \right) \mathrm{e}^{\mathrm{i}px} \frac{\mathrm{d}^2 p}{(2\pi)^2} &= \int_0^\infty \int_0^{2\pi} \ln\left(\frac{|p|}{2\mu} \mathrm{e}^\gamma \right) \mathrm{e}^{\mathrm{i}|p||x| \cos\phi} \frac{\cos\phi}{(2\pi)^2 |x|} \,\mathrm{d}\phi \,\mathrm{d}|p| \\ &= \frac{\mathrm{i}}{2\pi|x|} \int_0^\infty \ln\left(\frac{|p|}{2\mu} \mathrm{e}^\gamma \right) \mathrm{J}_1(|p||x|) \,\mathrm{d}|p| \\ &= \frac{\mathrm{i}}{2\pi|x|^2} \lim_{\delta\to 0} \int_0^\infty \frac{\left(\frac{t}{2\mu|x|} \mathrm{e}^\gamma \right)^{2\delta} - 1}{2\delta} \mathrm{J}_1(t) \,\mathrm{d}t \\ &= \frac{\mathrm{i}}{2\pi|x|^2} \lim_{\delta\to 0} \frac{\left(\frac{\mathrm{e}^\gamma}{\mu|x|} \right)^{2\delta} \frac{\Gamma(1+\delta)}{\Gamma(1-\delta)} - 1}{2\delta} \\ &= - \frac{\mathrm{i}}{2\pi|x|^2} \ln(\mu|x|) \,, \end{aligned} \tag{49}$$

where we used the integrals [46, Eqs. (10.9.2) and (10.22.43)]. We thus obtain the renormalised distribution $u^\text{ren}_{\mu\nu}$, acting on a test function as

$$u^\text{ren}_{\mu\nu}(f) = - \frac{1}{4} \int \ln\!\left(\mu^2 x^2\right) \left[\partial_\mu \partial_\nu f(x) + \delta_{\mu\nu} \frac{x^\rho}{x^2} \partial_\rho f(x) \right] \mathrm{d}^2 x \,, \tag{50}$$

and the divergent part is given by

$$u^\text{div}_{\mu\nu}(f) = - \pi \delta_{\mu\nu} \ln(\mu\epsilon) f(0) \,. \tag{51}$$

The negative of $u^\text{div}_{\mu\nu}$ is the required counterterm, which is local and diverges logarithmically with the UV cutoff ϵ as required.

To obtain the renormalised expectation value, we separate the terms with $j = k$ and $j \neq k$ in the last sum in the unrenormalised expectation value (40), and then replace $u_{\mu\nu}$ with $u^\text{ren}_{\mu\nu}$ in the terms with $j = k$. Taking the limit $\epsilon \to 0$, we obtain

$$\begin{aligned} &\left\langle \mathcal{N}_\mu[\mathcal{O}_{\mu\nu}(z)] \prod_{j=1}^n \mathcal{N}_\mu\left[V_{\sigma_j \beta}(x_j) \right] \right\rangle^{\Lambda,0}_{0,\text{ren}} \\ &= - \frac{\beta^2}{4\pi^2} \left(\frac{\Lambda}{\mu} \right)^{\frac{\beta^2}{4\pi} \left(\sum_{j=1}^n \sigma_j \right)^2} \prod_{1 \leq j < k \leq n} \left[\mu^2 (x_j - x_k)^2 \right]^{\sigma_j \sigma_k \frac{\beta^2}{4\pi}} \\ &\quad \times \left[\sum_{k=1}^n u^\text{ren}_{\mu\nu}(x_k - z) + 2 \sum_{1 \leq j < k \leq n} \sigma_j \sigma_k \frac{(x_j - z)_{(\mu}}{(x_j - z)^2} \frac{(x_k - z)_{\nu)}}{(x_k - z)^2} \right] , \end{aligned} \tag{52}$$

where $u^\text{ren}_{\mu\nu}(x)$ is the formal integral kernel of the distribution defined by equation (50). We see that taking the IR cutoff $\Lambda \to 0$, we have a non-vanishing result only if the sum of all σ_i vanishes, which is the super-selection criterion or neutrality condition of the vacuum sector [45].

For the renormalised expectation value of the stress tensor $T_{\mu\nu} = \mathcal{O}_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}\mathcal{O}_\rho{}^\rho + g\delta_{\mu\nu}(V_\beta + V_{-\beta})$, we obtain a sum of four terms, the first two of which are obtained from equation (52). Since the divergent part of $u_{\mu\nu}$ (51) is proportional to $\delta_{\mu\nu}$, it cancels out between the first two terms, i.e., we have

$$\begin{aligned}&\left\langle \mathcal{N}_\mu \left[\mathcal{O}_{\mu\nu}(z) - \frac{1}{2}\delta_{\mu\nu}\mathcal{O}_\rho{}^\rho(z) \right] \prod_{j=1}^n \mathcal{N}_\mu \left[V_{\sigma_j\beta}(x_j) \right] \right\rangle_{0,\mathrm{ren}}^{\Lambda,0} \\ &\quad = \lim_{\epsilon\to 0} \left\langle \mathcal{N}_\mu \left[\mathcal{O}_{\mu\nu}(z) - \frac{1}{2}\delta_{\mu\nu}\mathcal{O}_\rho{}^\rho(z) \right] \prod_{j=1}^n \mathcal{N}_\mu \left[V_{\sigma_j\beta}(x_j) \right] \right\rangle_0^{\Lambda,\epsilon}\end{aligned} \tag{53}$$

and no counterterm is actually necessary for this combination. For the last two terms, we compute

$$\begin{aligned}&\left\langle \mathcal{N}_\mu [V_\beta(z)] \prod_{j=1}^n \mathcal{N}_\mu \left[V_{\sigma_j\beta}(x_j) \right] \right\rangle_0^{\Lambda,\epsilon} = (\mu\epsilon)^{-(n+1)\frac{\beta^2}{4\pi}} \\ &\qquad \times \exp\left[-\frac{\beta^2}{2} C^{\Lambda,\epsilon}(z,z) - \beta^2 \sum_{j=1}^n \sigma_j C^{\Lambda,\epsilon}(z,x_j) - \frac{\beta^2}{2} \sum_{j,k=1}^n \sigma_j\sigma_k C^{\Lambda,\epsilon}(x_j,x_k) \right] \\ &\quad = \left(\frac{\Lambda}{\mu}\right)^{\frac{\beta^2}{4\pi}\left(1+\sum_{j=1}^n \sigma_j\right)^2} \prod_{j=1}^n \left[\mu^2 \left[(z-x_j)^2 + \epsilon^2 \right] \right]^{\sigma_j \frac{\beta^2}{4\pi}} \\ &\qquad \times \prod_{1\le j<k\le n} \left[\mu^2 \left[(x_j-x_k)^2 + \epsilon^2 \right] \right]^{\sigma_j\sigma_k \frac{\beta^2}{4\pi}} ,\end{aligned} \tag{54}$$

using the normal-ordering of vertex operators (26), equation (18) with $J(x) = \beta\delta^2(x-z) + \sum_{j=1}^n \sigma_j\beta\delta^2(x-x_j)$, and the explicit form of the covariance (23). Since we are in the finite regime $\beta^2 < 4\pi$, the singularities that arise for $\epsilon = 0$ as $x_j \to x_k$ and $x_j \to z$ are integrable, and so for this term no further renormalisation beyond the normal-ordering is required. Moreover, we again see how the neutrality condition appears: as $\Lambda \to 0$, we obtain a vanishing result unless $\sum_{j=1}^n \sigma_j = -1$. The last term with $V_{-\beta}$ results in the same result with σ_j replaced by $-\sigma_j$ on the right-hand side. □

2.3. Proof of theorem 2 (Convergence). In this whole section, we tacitly employ Fubini's theorem to interchange absolutely convergent integrals. We consider numerator and denominator of equation (4) separately. We start with the denominator, and compute analogously to equation (54) that

$$\begin{aligned}\left\langle \prod_{j=1}^n \mathcal{N}_\mu \left[V_{\sigma_j\beta}(x_j) \right] \right\rangle_{0,\mathrm{ren}}^{0,0} &= \lim_{\Lambda,\epsilon\to 0} \left\langle \prod_{j=1}^n \mathcal{N}_\mu \left[V_{\sigma_j\beta}(x_j) \right] \right\rangle_0^{\Lambda,\epsilon} \\ &= \delta_{0,\sum_{j=1}^n \sigma_j} \prod_{1\le j<k\le n} \left[\mu^2 (x_j-x_k)^2 \right]^{\sigma_j\sigma_k \frac{\beta^2}{4\pi}} .\end{aligned} \tag{55}$$

Since $\sigma_j = \pm 1$, to obtain a non-vanishing result we must have $n = 2m$ with m positive σ_j and m negative ones. We then rename the x_j with $\sigma_j = -1$ to y_j and renumber them. Taking into account that there are $\binom{n}{m} = (2m)!/(m!)^2$ possibilities to choose m positive σ_j from a total of $n = 2m$ ones (since equation (55) is symmetric under a permutation of the (renamed) x_i and y_j among themselves), the denominator of equation (4) reduces to

$$\begin{aligned} &\sum_{m=0}^{\infty} \frac{1}{(m!)^2} \int \cdots \int \left\langle \prod_{j=1}^{m} \mathcal{N}_\mu[V_\beta(x_j)]\, \mathcal{N}_\mu[V_{-\beta}(y_j)] \right\rangle_{0,\text{ren}}^{0,0} \prod_{i=1}^{m} g(x_i) g(y_i) \,\mathrm{d}^2 x_i \,\mathrm{d}^2 y_i \\ &= \sum_{m=0}^{\infty} \frac{\mu^{-m\frac{\beta^2}{2\pi}}}{(m!)^2} \int \cdots \int \left[\frac{\prod_{1 \leq j < k \leq m} (x_j - x_k)^2 (y_j - y_k)^2}{\prod_{j,k=1}^{m} (x_j - y_k)^2} \right]^{\frac{\beta^2}{4\pi}} \prod_{i=1}^{m} g(x_i) g(y_i) \,\mathrm{d}^2 x_i \,\mathrm{d}^2 y_i \,. \end{aligned} \tag{56}$$

As in previous works [1,47], to bound the term at order $2m$ we introduce complex variables $\chi_j \equiv x_j^1 + \mathrm{i} x_j^2$, $\upsilon_j \equiv y_j^1 + \mathrm{i} y_j^2$ such that

$$(x_i - y_j)^2 = (x_i^1 - y_j^1)^2 + (x_i^2 - y_j^2)^2 = \left| \chi_i - \upsilon_j \right|^2 \tag{57}$$

and analogously for $(x_i - x_j)^2 = \left| \chi_i - \chi_j \right|^2$ and $(y_i - y_j)^2 = \left| \upsilon_i - \upsilon_j \right|^2$. It follows that

$$\begin{aligned} &\left[\frac{\prod_{1 \leq i < j \leq n} (x_i - x_j)^2 \prod_{1 \leq i < j \leq n} (y_i - y_j)^2}{\prod_{i,j=1}^{n} (x_i - y_j)^2} \right]^p \\ &= \left| \frac{\prod_{1 \leq i < j \leq n} (\chi_i - \chi_j) \prod_{1 \leq i < j \leq n} (\upsilon_i - \upsilon_j)}{\prod_{i,j=1}^{n} (\chi_i - \upsilon_j)} \right|^{2p} = \left| \det \left(\frac{1}{\chi_i - \upsilon_j} \right)_{i,j=1}^{n} \right|^{2p} , \end{aligned} \tag{58}$$

where the last equality is the well-known Cauchy determinant formula [48]. We estimate

$$\left| \det \left(\frac{1}{\chi_i - \upsilon_j} \right)_{i,j=1}^{n} \right| \leq \sum_{\pi} \prod_{j=1}^{n} \frac{1}{\left| \chi_j - \upsilon_{\pi(j)} \right|} \,, \tag{59}$$

where the sum runs over all permutations π of $\{1, \ldots, n\}$. Using the inequality

$$\left(\sum_{j=1}^{k} \left| a_k \right| \right)^p \leq \begin{Bmatrix} k^{p-1} & p \geq 1 \\ 1 & 0 < p < 1 \end{Bmatrix} \sum_{j=1}^{k} \left| a_k \right|^p \leq \max\!\left(k^{p-1}, 1 \right) \sum_{j=1}^{k} \left| a_k \right|^p \,, \tag{60}$$

it follows that

$$\begin{aligned} &\left[\frac{\prod_{1 \leq i < j \leq n} (x_i - x_j)^2 \prod_{1 \leq i < j \leq n} (y_i - y_j)^2}{\prod_{i,j=1}^{n} (x_i - y_j)^2} \right]^p \leq \left[\sum_{\pi} \prod_{j=1}^{n} \frac{1}{\left| \chi_j - \upsilon_{\pi(j)} \right|} \right]^{2p} \\ &\qquad \leq \max\!\left((n!)^{2p-1}, 1 \right) \sum_{\pi} \prod_{j=1}^{n} \left| \chi_j - \upsilon_{\pi(j)} \right|^{-2p} \\ &\qquad = (n!)^{\max(2p-1,0)} \sum_{\pi} \prod_{j=1}^{n} \left[(x_j - y_{\pi(j)})^2 \right]^{-p} , \end{aligned} \tag{61}$$

since the number of permutations π of n elements is $n!$.

Inserting the estimate (61) with $p = \beta^2/(4\pi)$ into equation (56), the denominator of equation (4) is estimated by (we recall that $g \geq 0$)

$$\sum_{m=0}^{\infty} \frac{1}{(m!)^\gamma} \int \cdots \int \prod_{j=1}^{m} \left[\mu^2 (x_j - y_j)^2\right]^{-\frac{\beta^2}{4\pi}} \prod_{i=1}^{m} g(x_i) g(y_i) \, \mathrm{d}^2 x_i \, \mathrm{d}^2 y_i \tag{62}$$

with $\gamma \equiv 1 - \max(\beta^2/(2\pi) - 1, 0)$, and where an additional factor of $m!$ arose since all $m!$ permutations of the y_j give the same contribution. We see that we only need to bound

$$\iint \left[\mu^2 (x-y)^2\right]^{-\frac{\beta^2}{4\pi}} g(x) g(y) \, \mathrm{d}^2 x \, \mathrm{d}^2 y \,, \tag{63}$$

which we split in two parts: the region where $\mu^2 (x-y)^2 > 1$ and which we bound by $\iint g(x) g(y) \, \mathrm{d}^2 x \, \mathrm{d}^2 y = \|g\|_1^2$, and the region where $\mu^2 (x-y)^2 \leq 1$ and which we bound using Young's inequality. For this, we use it in the form [49]

$$|(f, g * h)| = \left| \iint f(x) g(x-y) h(y) \, \mathrm{d}^2 x \, \mathrm{d}^2 y \right| \leq \|f\|_p \|g\|_q \|h\|_r \,, \tag{64}$$

where $p, q, r \geq 1$ with $1/p + 1/q + 1/r = 2$. Taking

$$q = 1 + \frac{4\pi - \beta^2}{8\pi} \,, \quad p = r = 1 + \frac{4\pi}{8\pi - \beta^2} \,, \tag{65}$$

the condition $p, q, r \geq 1$ is fulfilled since we are in the finite regime $\beta^2 < 4\pi$, as well as $p, q, r < 2$. We obtain

$$\iint_{\mu|x-y| \leq 1} \left[\mu^2 (x-y)^2\right]^{-\frac{\beta^2}{4\pi}} g(x) g(y) \, \mathrm{d}^2 x \, \mathrm{d}^2 y \leq \|g\|_p^2 \left\| \Theta(1 - \mu|x|) (\mu|x|)^{-\frac{\beta^2}{2\pi}} \right\|_q \tag{66}$$

and

$$\begin{aligned} \left\| \Theta(1 - \mu|x|) (\mu|x|)^{-\frac{\beta^2}{2\pi}} \right\|_q^q &= \int_{\mu|x| \leq 1} (\mu|x|)^{-\frac{\beta^2}{2\pi} q} \, \mathrm{d}^2 x \\ &= \frac{32\pi^3}{(4\pi - \beta^2)(8\pi - \beta^2)} \mu^{-2} \,, \end{aligned} \tag{67}$$

where q (65) was chosen in such a way that the integral is finite. Summing the contributions from both regions, we have shown that there exists a constant K (depending on β and g) such that

$$\iint \left[\mu^2 (x-y)^2\right]^{-\frac{\beta^2}{4\pi}} g(x) g(y) \, \mathrm{d}^2 x \, \mathrm{d}^2 y \leq K \,, \tag{68}$$

and hence the denominator (56) is bounded from above by

$$\sum_{m=0}^{\infty} K^m \begin{Bmatrix} (m!)^{-1} & \beta^2 < 2\pi \\ (m!)^{\frac{\beta^2}{2\pi} - 2} & 2\pi \leq \beta^2 < 4\pi \end{Bmatrix} < \infty \,. \tag{69}$$

Moreover, since each term in the sum (56) is positive, it is bounded from below by the first term which is 1. Let us remark that the bounds (69) are not new and known from [1]; we have repeated their derivation to make the proof self-contained, and since very similar estimates are needed for the numerator.

Consider thus the numerator of equation (4), where using the result (52) for the renormalised expectation values again only even terms with $n = 2m$ contribute. Taking into account the symmetry under the exchange of variables and renaming integration variables as in the case of the denominator, the numerator reads

$$\begin{aligned} &- \frac{\beta^2}{4\pi^2} \sum_{m=0}^{\infty} \frac{\mu^{-m\frac{\beta^2}{2\pi}}}{(m!)^2} \int f(z) \int \cdots \int \left[\frac{\prod_{1 \leq j < k \leq m} (x_j - x_k)^2 (y_j - y_k)^2}{\prod_{j,k=1}^{m} (x_j - y_k)^2} \right]^{\frac{\beta^2}{4\pi}} \\ &\times \Bigg[\sum_{k=1}^{m} u^{\text{ren}}_{\mu\nu}(x_k - z) + \sum_{k=1}^{m} u^{\text{ren}}_{\mu\nu}(y_k - z) + 2 \sum_{1 \leq j < k \leq m} \frac{(x_j - z)_{(\mu}}{(x_j - z)^2} \frac{(x_k - z)_{\nu)}}{(x_k - z)^2} \\ &\qquad - 2 \sum_{j,k=1}^{m} \frac{(x_j - z)_{(\mu}}{(x_j - z)^2} \frac{(y_k - z)_{\nu)}}{(y_k - z)^2} + 2 \sum_{1 \leq j < k \leq m} \frac{(y_j - z)_{(\mu}}{(y_j - z)^2} \frac{(y_k - z)_{\nu)}}{(y_k - z)^2} \Bigg] \\ &\times \mathrm{d}^2 z \prod_{i=1}^{m} g(x_i) g(y_i) \, \mathrm{d}^2 x_i \, \mathrm{d}^2 y_i \,. \end{aligned} \tag{70}$$

Since we can interchange x_j and y_j without changing the result, we see that there are three different types of terms: the ones involving the renormalised $u^{\text{ren}}_{\mu\nu}$, the ones involving a double sum and only x_j, and the ones with a double sum over both x_j and y_j. We start with the last two types, which after the shift $x_j \to x_j + z$, $y_j \to y_j + z$ and with the estimate (61) with $p = \beta^2/(4\pi)$ are bounded by

$$\begin{aligned} &\frac{\beta^2}{4\pi^2} \sum_{m=0}^{\infty} \frac{1}{(m!)^{1+\gamma}} \int |f(z)| \int \cdots \int \sum_{\pi} \prod_{j=1}^{m} \left[\mu^2 (x_j - y_{\pi(j)})^2 \right]^{-\frac{\beta^2}{4\pi}} \\ &\times \left[4 \sum_{1 \leq j < k \leq m} \frac{1}{|x_j||x_k|} + 2 \sum_{j,k=1}^{m} \frac{1}{|x_j||y_k|} \right] \mathrm{d}^2 z \prod_{i=1}^{m} g(x_i + z) g(y_i + z) \, \mathrm{d}^2 x_i \, \mathrm{d}^2 y_i \end{aligned} \tag{71}$$

with the same γ as before, defined after equation (62). Since the remainder of the integrand is invariant under permutations of the y_j, the sum over permutations π coming from the determinant estimate again gives a factor of $m!$. We then use again Young's inequality (64), separating contributions involving a $1/|x_k|$ or $1/|y_k|$ from the sums from the ones without. The contributions without such terms are estimated as before (68), taking into account that the norm of g with shifted argument $x + z$ is equal to the norm of g because the integral that defines the norm is translation invariant. For the other contributions, we begin with the first sum, where a single term $1/|x_j|$ may be present. In the region where $\mu|x_j - y_j| > 1$, we then estimate

$$\begin{aligned}
&\iint_{\mu|x_j-y_j|>1} \left[\mu^2(x_j-y_j)^2\right]^{-\frac{\beta^2}{4\pi}} \frac{1}{|x_j|} g(x_j+z)g(y_j+z)\,\mathrm{d}^2x_j\,\mathrm{d}^2y_j \\
&\quad \leq \|g\|_1 \int \frac{1}{|x_j|} g(x_j+z)\,\mathrm{d}^2x_j \\
&\quad = \|g\|_1 \int \frac{1}{|x_j|(\mu^2x_j^2+1)} g(x_j+z)(\mu^2x_j^2+1)\,\mathrm{d}^2x_j \\
&\quad \leq \pi^2\mu^{-1}\|g\|_1 \left\|(\mu^2x^2+1)g(x+z)\right\|_\infty
\end{aligned} \tag{72}$$

using Hölder's inequality

$$\|fg\|_1 \leq \|f\|_{r/(r-1)}\|g\|_r \tag{73}$$

with $r = 1$. In the region where $\mu|x_j - y_j| \leq 1$, we use instead Young's inequality (64) with the same choice of exponents (65) and obtain

$$\begin{aligned}
&\iint_{\mu|x_j-y_j|\leq 1} \left[\mu^2(x_j-y_j)^2\right]^{-\frac{\beta^2}{4\pi}} \frac{1}{|x_j|} g(x_j+z)g(y_j+z)\,\mathrm{d}^2x_j\,\mathrm{d}^2y_j \\
&\quad \leq \|g\|_p \left\|\frac{1}{|x|}g(x+z)\right\|_p \left\|\Theta(1-\mu|x|)(\mu|x|)^{-\frac{\beta^2}{2\pi}}\right\|_q
\end{aligned} \tag{74}$$

and, using again Hölder's inequality (73) with $r = 1$,

$$\begin{aligned}
\left\|\frac{1}{|x|}g(x+z)\right\|_p^p &= \int \frac{1}{|x|^p(\mu^2x^2+1)} g^p(x+z)(\mu^2x^2+1)\,\mathrm{d}^2x \\
&\leq \pi\mu^{p-2}\Gamma\left(1-\frac{p}{2}\right)\Gamma\left(\frac{p}{2}\right)\left\|(\mu^2x^2+1)g^p(x+z)\right\|_\infty < \infty\,,
\end{aligned} \tag{75}$$

since the choice (65) is such that $1 < p < 2$. Thus also in this case there exists a constant K (depending on β and g) such that

$$\iint \left[\mu^2(x_j-y_j)^2\right]^{-\frac{\beta^2}{4\pi}} \frac{1}{|x_j|} g(x_j+z)g(y_j+z)\,\mathrm{d}^2x_j\,\mathrm{d}^2y_j \leq K\,. \tag{76}$$

If the second sum involves $\frac{1}{|x_j||y_k|}$ with $j \neq k$, we have the same estimates, while for $j = k$ we obtain analogously to the above

$$\begin{aligned}
&\iint_{\mu|x_j-y_j|>1} \left[\mu^2(x_j-y_j)^2\right]^{-\frac{\beta^2}{4\pi}} \frac{1}{|x_j||y_j|} g(x_j+z)g(y_j+z)\,\mathrm{d}^2x_j\,\mathrm{d}^2y_j \\
&\quad \leq \pi^4\mu^{-2}\left\|(\mu^2x^2+1)g(x+z)\right\|_\infty^2
\end{aligned} \tag{77}$$

and

$$\begin{aligned}
&\iint_{\mu|x_j-y_j|\leq 1} \left[\mu^2(x_j-y_j)^2\right]^{-\frac{\beta^2}{4\pi}} \frac{1}{|x_j|} g(x_j+z)g(y_j+z)\,\mathrm{d}^2x_j\,\mathrm{d}^2y_j \\
&\quad \leq \left\|\frac{1}{|x|}g(x+z)\right\|_p^2 \left\|\Theta(1-\mu|x|)(\mu|x|)^{-\frac{\beta^2}{2\pi}}\right\|_q\,,
\end{aligned} \tag{78}$$

and so also in this case we can bound everything by a constant K depending on β and g. It follows that equation (71) is bounded by

$$\begin{aligned} &\frac{\beta^2}{4\pi^2} \sum_{m=0}^\infty \frac{1}{(m!)^\gamma} \int \lvert f(z) \rvert \left[4 \sum_{1 \leq j < k \leq m} K^m + 2 \sum_{j,k=1}^m K^m \right] \mathrm{d}^2 z \\ &\quad\leq C \sum_{m=0}^\infty \frac{1}{(m!)^\gamma} m^2 K^m < \infty \end{aligned} \tag{79}$$

with another constant $C = 4/\pi \lVert f \rVert_1$.

For the remaining terms in the numerator (70) involving the renormalised $u^\text{ren}_{\mu\nu}$, we recall its definition (50) from which it follows that

$$\begin{aligned} &\int u^\text{ren}_{\mu\nu}(x-z) f(z) \,\mathrm{d}^2 z \\ &\quad= - \frac{1}{4} \int \ln\!\big[\mu^2 (x-z)^2 \big] \left[\partial_\mu \partial_\nu f(z) - \delta_{\mu\nu} \frac{(x-z)^\rho}{(x-z)^2} \partial_\rho f(z) \right] \mathrm{d}^2 z \,. \end{aligned} \tag{80}$$

We then again shift $x_j \to x_j + z$, $y_j \to y_j + z$ and use the determinant estimate (61) with $p = \beta^2/(4\pi)$ to obtain the bound

$$\begin{aligned} &\frac{\beta^2}{4\pi^2} \sum_{m=0}^\infty \frac{1}{(m!)^{1+\gamma}} \int \int \cdots \int \sum_\pi \prod_{j=1}^m \big[\mu^2 (x_j - y_{\pi(j)})^2 \big]^{-\frac{\beta^2}{4\pi}} \\ &\quad\times \sum_{k=1}^m \lvert \ln(\mu \lvert x_k \rvert) \rvert \left[\sup_{\mu,\nu \in \{1,2\}} \lvert \partial_\mu \partial_\nu f(z) \rvert + \frac{1}{\lvert x_k \rvert} \sup_{\rho \in \{1,2\}} \lvert \partial_\rho f(z) \rvert \right] \\ &\quad\times \mathrm{d}^2 z \prod_{i=1}^m g(x_i + z) g(y_i + z) \,\mathrm{d}^2 x_i \,\mathrm{d}^2 y_i \end{aligned} \tag{81}$$

with the same γ as before. The sum over permutations π again gives a factor $m!$, and for the terms with the x_k not involved in the sum, we have the same bounds (68) as before. For the other terms, in the region where $\mu \lvert x_k - y_k \rvert > 1$ we estimate that

$$\begin{aligned} &\iint_{\mu \lvert x_k - y_k \rvert > 1} \big[\mu^2 (x_k - y_k)^2 \big]^{-\frac{\beta^2}{4\pi}} \lvert \ln(\mu \lvert x_k \rvert) \rvert g(x_k + z) g(y_k + z) \,\mathrm{d}^2 x_k \,\mathrm{d}^2 y_k \\ &\leq \lVert g \rVert_1 \int \lvert \ln(\mu \lvert x_k \rvert) \rvert g(x_k + z) \,\mathrm{d}^2 x_k \\ &= \lVert g \rVert_1 \int \frac{\lvert \ln(\mu \lvert x_k \rvert) \rvert}{(\mu^2 x_k^2 + 1)^2} g(x_k + z) (\mu^2 x_k^2 + 1)^2 \,\mathrm{d}^2 x_k \\ &\leq \frac{\pi}{\mu^2} \ln 2 \lVert g \rVert_1 \big\lVert (\mu^2 x^2 + 1)^2 g(x+z) \big\rVert_\infty \,, \end{aligned} \tag{82}$$

using again Hölder's inequality (73) with $r = 1$, and analogously

$$\begin{aligned} &\iint_{\mu \lvert x_k - y_k \rvert > 1} \big[\mu^2 (x_k - y_k)^2 \big]^{-\frac{\beta^2}{4\pi}} \frac{\lvert \ln(\mu \lvert x_k \rvert) \rvert}{\lvert x_k \rvert} g(x_k + z) g(y_k + z) \,\mathrm{d}^2 x_k \,\mathrm{d}^2 y_k \\ &\leq \frac{4\pi}{\mu} \lVert g \rVert_1 \big\lVert (\mu^2 x^2 + 1) g(x+z) \big\rVert_\infty \,. \end{aligned} \tag{83}$$

In the region where $\mu|x_k - y_k| \leq 1$, we use again Young's inequality (64) with the same choice of exponents (65) and obtain

$$\begin{aligned}
&\iint_{\mu|x_k-y_k|\leq 1} \left[\mu^2(x_k-y_k)^2\right]^{-\frac{\beta^2}{4\pi}} |\ln(\mu|x_k|)| g(x_k+z) g(y_k+z)\, \mathrm{d}^2x_k\, \mathrm{d}^2y_k \\
&\quad \leq \|g\|_p \| |\ln(\mu|x|)| g(x+z)\|_p \left\| \Theta(1-\mu|x|)(\mu|x|)^{-\frac{\beta^2}{2\pi}} \right\|_q
\end{aligned} \tag{84}$$

and, using again Hölder's inequality (73) with $r = 1$,

$$\begin{aligned}
\| |\ln(\mu|x|)| g(x+z)\|_p^p &= \int \frac{|\ln(\mu|x|)|^p}{(\mu^2x^2+1)^2} g^p(x+z)(\mu^2x^2+1)^2\, \mathrm{d}^2x \\
&\leq 6\mu^{-2} \left\| (\mu^2x^2+1)^2 g^p(x+z) \right\|_\infty < \infty\,,
\end{aligned} \tag{85}$$

and analogously

$$\begin{aligned}
&\iint_{\mu|x_k-y_k|\leq 1} \left[\mu^2(x_k-y_k)^2\right]^{-\frac{\beta^2}{4\pi}} \frac{|\ln(\mu|x_k|)|}{|x_k|} g(x_k+z) g(y_k+z)\, \mathrm{d}^2x_k\, \mathrm{d}^2y_k \\
&\quad \leq \|g\|_p \left\| \frac{|\ln(\mu|x|)|}{|x|} g(x+z) \right\|_p \left\| \Theta(1-\mu|x|)(\mu|x|)^{-\frac{\beta^2}{2\pi}} \right\|_q
\end{aligned} \tag{86}$$

with

$$\begin{aligned}
\left\| \frac{|\ln(\mu|x|)|}{|x|} g(x+z) \right\|_p^p &= \int \frac{|\ln(\mu|x|)|^p}{|x|(\mu^2x^2+1)^2} g^p(x+z)(\mu^2x^2+1)^2\, \mathrm{d}^2x \\
&\leq 25\mu^{-2} \left\| (\mu^2x^2+1)^2 g^p(x+z) \right\|_\infty < \infty\,.
\end{aligned} \tag{87}$$

It follows that there exists a constant K (depending on β and g) such that equation (81) is bounded by

$$C \sum_{m=0}^{\infty} \frac{1}{(m!)^\gamma} m K^m < \infty \tag{88}$$

with another constant $C = \frac{1}{\pi}\left[\left\| \sup_{\mu,\nu\in\{1,2\}} |\partial_\mu \partial_\nu f(z)| \right\|_1 + \left\| \sup_{\rho\in\{1,2\}} |\partial_\rho f(z)| \right\|_1 \right]$.

Taking all together, the numerator of equation (4) is bounded by (5), with K being the maximum of all the constants K in this section, and C being the sum of all the constants C in this section. Since the denominator is bounded from below by 1, the bound (5) holds for the full Gell-Mann–Low expectation value (4).

For the first two terms of the stress tensor $T_{\mu\nu} = \mathcal{O}_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}\mathcal{O}_\rho{}^\rho + g\delta_{\mu\nu}(V_\beta + V_{-\beta})$ we can take over the above bounds. For the third term, we use the result (54) and thus have to bound

$$\begin{aligned}
&\sum_{n=0}^\infty \frac{1}{n!} \int \cdots \int \sum_{\sigma_i = \pm 1} \left\langle \mathcal{N}_\mu[V_\beta(gf)] \prod_{j=1}^n \mathcal{N}_\mu\left[V_{\sigma_j \beta}(x_j)\right] \right\rangle_{0,\text{ren}}^{0,0} \prod_{i=1}^n g(x_i) \,\mathrm{d}^2 x_i \\
&= \sum_{m=0}^\infty \frac{\mu^{-(m+1)\frac{\beta^2}{2\pi}}}{m!(m+1)!} \int g(z) f(z) \int \cdots \int \prod_{j=1}^m \left[\frac{(z-x_j)^2}{(z-y_j)^2} \right]^{\frac{\beta^2}{4\pi}} \left[\frac{1}{(z-y_{m+1})^2} \right]^{\frac{\beta^2}{4\pi}} \\
&\quad \times \left[\frac{\prod_{1 \leq j < k \leq m} (x_j - x_k)^2 (y_j - y_k)^2}{\prod_{j,k=1}^m (x_j - y_k)^2} \prod_{j=1}^m \frac{(y_j - y_{m+1})^2}{(x_j - y_{m+1})^2} \right]^{\frac{\beta^2}{4\pi}} \\
&\quad \times \mathrm{d}^2 z \prod_{i=1}^m g(x_i) \,\mathrm{d}^2 x_i \prod_{j=1}^{m+1} g(y_j) \,\mathrm{d}^2 y_j \,,
\end{aligned} \tag{89}$$

where we used that because of the neutrality condition only odd terms $n = 2m+1$ give a non-vanishing contribution. Of these, m have a positive σ_j and $m+1$ have a negative one, such that the sum over the σ_j resulted in a factor of $\binom{2m+1}{m} = (2m+1)!/(m!(m+1)!)$, and as before we renamed the integration variables with a negative σ_j to y_j. Setting $x_{m+1} \equiv z$, the terms in brackets combine to the expression (58) with $n = m+1$ and $p = \beta^2/(4\pi)$, and we can use the Cauchy determinant formula and the bound (61) for the determinant. We can thus bound the series (89) by

$$\begin{aligned}
&\sum_{m=0}^\infty \frac{1}{m![(m+1)!]^\gamma} \int g(z) |f(z)| \int \cdots \int \sum_\pi \left[\mu^2 (z - y_{\pi(m+1)})^2 \right]^{-\frac{\beta^2}{4\pi}} \\
&\quad \times \prod_{j=1}^m \left[\mu^2 (x_j - y_{\pi(j)})^2 \right]^{-\frac{\beta^2}{4\pi}} \mathrm{d}^2 z \prod_{i=1}^m g(x_i) \,\mathrm{d}^2 x_i \prod_{j=1}^{m+1} g(y_j) \,\mathrm{d}^2 y_j \,,
\end{aligned} \tag{90}$$

and the sum over permutations π gives a factor $(m+1)!$ since the remainder of the integrand is symmetric under the interchange of the y_j. We can now use the bound (68) (estimating together the integrals over z and y_{m+1}), and obtain for the series (89) the bound

$$C \sum_{m=0}^\infty \frac{1}{[(m+1)!]^\gamma} (m+1) K^m \leq C \sum_{m=0}^\infty \frac{1}{(m!)^\gamma} m^2 K^m < \infty \,, \tag{91}$$

with C now also depending on g (from the integral over y_{m+1} and z). The same bound is obtained analogously for the fourth term in the stress tensor involving $V_{-\beta}$, which switches x_i with y_i in equation (89), such that taking all together the bound (5) holds also for the stress tensor. □

2.4. Proof of theorem 3 (Conservation). Since we have shown in the last subsection that the denominator of the Gell-Mann–Low formula (4) is bounded from below, to show that the interacting stress tensor is conserved it is enough to show that the numerator vanishes when smeared with a test function of the form $\partial^\mu f$

with $f \in \mathcal{S}(\mathbb{R}^2)$. Consider the numerator for $\mathcal{O}_{\mu\nu}$ (70) smeared with $\partial^\mu f$, which contains two different types of terms: the ones with the renormalised $u^{\mathrm{ren}}_{\mu\nu}$, and the others involving double sums. We start with the second type, and compute

$$\begin{aligned}
&\int \frac{(x-z)_{(\mu}}{(x-z)^2}\frac{(y-z)_{\nu)}}{(y-z)^2}\partial^\mu f(z)\,\mathrm{d}^2z \\
&= \frac{1}{4}\partial^x_{(\mu}\partial^y_{\nu)}\int \ln\big[\mu^2(x-z)^2\big]\ln\big[\mu^2(y-z)^2\big]\partial^\mu f(z)\,\mathrm{d}^2z \\
&= \frac{1}{8}\triangle_x\partial^y_\nu\int \ln\big[\mu^2(x-z)^2\big]\ln\big[\mu^2(y-z)^2\big]f(z)\,\mathrm{d}^2z \\
&\quad + \frac{1}{8}\triangle_y\partial^x_\nu\int \ln\big[\mu^2(x-z)^2\big]\ln\big[\mu^2(y-z)^2\big]f(z)\,\mathrm{d}^2z \\
&\quad + \frac{1}{8}\partial^x_\mu\partial^\mu_y(\partial^x_\nu+\partial^y_\nu)\int \ln\big[\mu^2(x-z)^2\big]\ln\big[\mu^2(y-z)^2\big]f(z)\,\mathrm{d}^2z \\
&= \frac{\pi}{2}\partial^y_\nu\ln\big[\mu^2(y-x)^2\big]f(x)+\frac{\pi}{2}\partial^x_\nu\ln\big[\mu^2(x-y)^2\big]f(y) \\
&\quad + \frac{1}{8}\partial^x_\mu\partial^\mu_y(\partial^x_\nu+\partial^y_\nu)\int \ln\big[\mu^2(x-z)^2\big]\ln\big[\mu^2(y-z)^2\big]f(z)\,\mathrm{d}^2z\,,
\end{aligned} \tag{92}$$

using that the logarithm is a fundamental solution of the Laplace equation in 2 dimensions:

$$\triangle\ln\big[\mu^2(x-z)^2\big] = 4\pi\delta^2(x-z)\,, \tag{93}$$

as follows for example from the limit $m \to 0$ of equation (24) for the massive covariance after taking derivatives. Analogously, we obtain

$$\begin{aligned}
&\int \frac{(x-z)_\rho}{(x-z)^2}\frac{(y-z)^\rho}{(y-z)^2}\partial_\nu f(z)\,\mathrm{d}^2z \\
&= \frac{1}{4}\partial^x_\rho\partial^\rho_y(\partial^x_\nu+\partial^y_\nu)\int \ln\big[\mu^2(x-z)^2\big]\ln\big[\mu^2(y-z)^2\big]f(z)\,\mathrm{d}^2z\,,
\end{aligned} \tag{94}$$

which cancels the last line of (92) if we take the combination $\mathcal{O}_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}\mathcal{O}_\rho{}^\rho$ that appears in the stress tensor. The terms of the second type in this combination thus sum up to

$$\begin{aligned}
&\frac{\beta^2}{4\pi}\sum_{m=0}^\infty \frac{\mu^{-m\frac{\beta^2}{2\pi}}}{(m!)^2}\int\cdots\int\left[\frac{\prod_{1\le j<k\le m}(x_j-x_k)^2(y_j-y_k)^2}{\prod_{j,k=1}^m(x_j-y_k)^2}\right]^{\frac{\beta^2}{4\pi}} \\
&\times\Bigg[\sum_{1\le j<k\le m}[f(x_j)-f(x_k)]\partial_\nu\ln\big[\mu^2(x_j-x_k)^2\big] \\
&\quad - \sum_{j,k=1}^m[f(x_j)-f(y_k)]\partial_\nu\ln\big[\mu^2(x_j-y_k)^2\big] \\
&\quad + \sum_{1\le j<k\le m}[f(y_j)-f(y_k)]\partial_\nu\ln\big[\mu^2(y_j-y_k)^2\big]\Bigg]\prod_{i=1}^m g(x_i)g(y_i)\,\mathrm{d}^2x_i\,\mathrm{d}^2y_i\,.
\end{aligned} \tag{95}$$

We compute

$$\begin{aligned} &\partial_\nu^{x_\ell} \ln\left(\left[\frac{\prod_{1\leq j<k\leq m} (x_j-x_k)^2 (y_j-y_k)^2}{\prod_{j,k=1}^m (x_j-y_k)^2} \right]^{\frac{\beta^2}{4\pi}} \right) \\ &\quad = \frac{\beta^2}{2\pi} \left[\sum_{k=1}^{\ell-1} \frac{(x_\ell-x_k)_\nu}{(x_\ell-x_k)^2} + \sum_{k=\ell+1}^{m} \frac{(x_\ell-x_k)_\nu}{(x_\ell-x_k)^2} - \sum_{k=1}^{m} \frac{(x_\ell-y_k)_\nu}{(x_\ell-y_k)^2} \right] , \end{aligned} \tag{96}$$

and the analogous equation with x and y exchanged, multiply by $f(x_\ell)$, sum over ℓ and rename summation indices to obtain

$$\begin{aligned} &\sum_{k=1}^m f(x_k) \partial_\nu^{x_k} \ln\left(\left[\frac{\prod_{1\leq j<k\leq m} (x_j-x_k)^2 (y_j-y_k)^2}{\prod_{j,k=1}^m (x_j-y_k)^2} \right]^{\frac{\beta^2}{4\pi}} \right) \\ &\quad = \frac{\beta^2}{2\pi} \left[\sum_{1\leq j<k\leq m} [f(x_j)-f(x_k)] \frac{(x_j-x_k)_\nu}{(x_j-x_k)^2} - \sum_{j,k=1}^{m} f(x_j) \frac{(x_j-y_k)_\nu}{(x_j-y_k)^2} \right] , \end{aligned} \tag{97}$$

as well as the analogous equation with x and y exchanged. We can thus rewrite equation (95) in the form

$$\begin{aligned} &\sum_{m=0}^\infty \frac{\mu^{-m\frac{\beta^2}{2\pi}}}{(m!)^2} \int\cdots\int \prod_{i=1}^m \mathrm{d}^2 x_i \, \mathrm{d}^2 y_i \, g(x_i) g(y_i) \\ &\quad \times \sum_{k=1}^m [f(x_k)\partial_\nu^{x_k} + f(y_k)\partial_\nu^{y_k}] \left[\frac{\prod_{1\leq j<k\leq m} (x_j-x_k)^2 (y_j-y_k)^2}{\prod_{j,k=1}^m (x_j-y_k)^2} \right]^{\frac{\beta^2}{4\pi}} , \end{aligned} \tag{98}$$

and since by assumption g is constant on the support of f, we can integrate the derivatives by parts such that they act on f and then use the symmetry of the integrand under the exchange of the x_i and the y_i, such that the sum over k gives a factor m. Since the term with $m = 0$ does not contribute, renaming the summation index $m \to m+1$ we thus obtain

$$\begin{aligned} &-\sum_{m=0}^\infty \frac{\mu^{-(m+1)\frac{\beta^2}{2\pi}}}{m!(m+1)!} \int\cdots\int \left[\frac{\prod_{1\leq j<k\leq m+1} (x_j-x_k)^2 (y_j-y_k)^2}{\prod_{j,k=1}^{m+1} (x_j-y_k)^2} \right]^{\frac{\beta^2}{4\pi}} \\ &\quad \times [\partial_\nu f(x_{m+1}) + \partial_\nu f(y_{m+1})] \prod_{i=1}^{m+1} g(x_i) g(y_i) \, \mathrm{d}^2 x_i \, \mathrm{d}^2 y_i \end{aligned} \tag{99}$$

for the terms of the second type in the combination $\mathcal{O}_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}\mathcal{O}_\rho{}^\rho$. This now cancels exactly the contribution from the last two terms of the stress tensor $T_{\mu\nu} = \mathcal{O}_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}\mathcal{O}_\rho{}^\rho + g\delta_{\mu\nu}(V_\beta + V_{-\beta})$ smeared with $\partial^\mu f$: this is seen by comparing equation (99) with the result (89) (renaming $z = x_{m+1}$ in that equation), and the analogous result for $V_{-\beta}$ which is obtained by exchanging x and y.

It follows that the numerator of the Gell-Mann–Low formula for $T_{\mu\nu}$ smeared with $\partial^\mu f$ only involves the terms with the renormalised $u^{\text{ren}}_{\mu\nu}$. For them, we use equation (80) and compute

$$\begin{aligned}&\int u^{\text{ren}}_{\mu\nu}(x-z)\partial^\mu f(z)\,\mathrm{d}^2z\\&= -\frac{1}{4}\int \ln\bigl[\mu^2(x-z)^2\bigr]\left[\triangle\partial_\nu f(z) - \frac{(x-z)^\rho}{(x-z)^2}\partial_\rho\partial_\nu f(z)\right]\mathrm{d}^2z\\&= -\pi\partial_\nu f(x) + \frac{1}{4}\int \ln\bigl[\mu^2(x-z)^2\bigr]\frac{(x-z)^\rho}{(x-z)^2}\partial_\rho\partial_\nu f(z)\,\mathrm{d}^2z\end{aligned} \tag{100}$$

as well as

$$\begin{aligned}&\int u_\rho{}^{\rho\,\text{ren}}(x-z)\partial_\nu f(z)\,\mathrm{d}^2z\\&= -\frac{1}{4}\int \ln\bigl[\mu^2(x-z)^2\bigr]\left[\triangle\partial_\nu f(z) - 2\frac{(x-z)^\rho}{(x-z)^2}\partial_\rho\partial_\nu f(z)\right]\mathrm{d}^2z\\&= -\pi\partial_\nu f(x) + \frac{1}{2}\int \ln\bigl[\mu^2(x-z)^2\bigr]\frac{(x-z)^\rho}{(x-z)^2}\partial_\rho\partial_\nu f(z)\,\mathrm{d}^2z\,,\end{aligned} \tag{101}$$

using again that the logarithm is a fundamental solution of the Laplace equation in two dimensions. In the combination $u^{\text{ren}}_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}u_\rho{}^{\rho\,\text{ren}}$ the integrals again cancel and we are left with the first terms:

$$\begin{aligned}&\sum_{n=0}^\infty \frac{1}{n!}\int\cdots\int \sum_{\sigma_i=\pm1}\left\langle \mathcal{N}_\mu[T_{\mu\nu}(\partial^\mu f)]\prod_{j=1}^n \mathcal{N}_\mu\bigl[V_{\sigma_j\beta}(x_j)\bigr]\right\rangle^{0,0}_{0,\text{ren}}\prod_{i=1}^n g(x_i)\,\mathrm{d}^2x_i\\&= \frac{\beta^2}{8\pi}\sum_{m=0}^\infty \frac{\mu^{-m\frac{\beta^2}{2\pi}}}{(m!)^2}\int\cdots\int\left[\frac{\prod_{1\le j<k\le m}(x_j-x_k)^2(y_j-y_k)^2}{\prod_{j,k=1}^m (x_j-y_k)^2}\right]^{\frac{\beta^2}{4\pi}}\\&\quad\times\sum_{k=1}^m\Bigl[\partial_\nu f(x_k)+\partial_\nu f(y_k)\Bigr]\prod_{i=1}^m g(x_i)g(y_i)\,\mathrm{d}^2x_i\,\mathrm{d}^2y_i\,.\end{aligned} \tag{102}$$

Comparing with the result (89) with $z = x_{m+1}$, and the analogous result for $V_{-\beta}$ which is obtained by exchanging x and y, it follows that

$$\sum_{n=0}^\infty \frac{1}{n!}\int\cdots\int \sum_{\sigma_i=\pm1}\left\langle \mathcal{N}_\mu\Bigl[\hat{T}_{\mu\nu}(\partial^\mu f)\Bigr]\prod_{j=1}^n \mathcal{N}_\mu\bigl[V_{\sigma_j\beta}(x_j)\bigr]\right\rangle^{0,0}_{0,\text{ren}}\prod_{i=1}^n g(x_i)\,\mathrm{d}^2x_i = 0 \tag{103}$$

and thus equation (6), where

$$\begin{aligned}\hat{T}_{\mu\nu} &\equiv T_{\mu\nu} - \frac{\beta^2}{8\pi}g\delta_{\mu\nu}(V_\beta+V_{-\beta})\\&= \mathcal{O}_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}\mathcal{O}_\rho{}^\rho + g\left(1-\frac{\beta^2}{8\pi}\right)\delta_{\mu\nu}(V_\beta+V_{-\beta})\end{aligned} \tag{104}$$

is the quantum-corrected stress tensor. □

3. Minkowski case

In the Minkowski case, we use the framework of perturbative algebraic quantum field theory (pAQFT), whose main advantage is the clean separation of algebraic issues (including renormalisation) from the construction of a state. Reviews of pAQFT can be found in [50–52], and we again take formulas and results without specifying their source explicitly.

3.1. Preliminaries. In pAQFT, one first constructs the algebra of free fields $\mathfrak{A}_0$ as the free algebra over smeared fields $\phi(f) = (f, \phi)$ with $f \in \mathcal{S}(\mathbb{R}^2)$ and their adjoints $[\phi(f)]^\dagger$ ($= \phi(f^*)$ for the real scalar field that we are considering), with unit $\mathbb{1}$ and the non-commutative product $\star$, modulo the commutation relations

$$[\phi(f), \phi(g)]_\star \equiv \phi(f) \star \phi(g) - \phi(g) \star \phi(f) = \mathrm{i}(f, \mathit{\Delta} * g)\mathbb{1} \,. \tag{105}$$

where the scalar product $(\cdot, \cdot)$ and convolution $*$ are defined in equations (19) and (20). $\mathit{\Delta}$ is the commutator function defined as the difference between retarded and advanced fundamental solutions of the Klein–Gordon equation

$$\mathit{\Delta}(x, y) \equiv G_\mathrm{ret}(x, y) - G_\mathrm{adv}(x, y) \,, \qquad \partial^2 G_\mathrm{ret}(x, y) = \delta^2(x - y) = \partial^2 G_\mathrm{adv}(x, y) \,, \tag{106}$$

which are unique in any globally hyperbolic spacetime, in particular Minkowski space. A state ω is given by a linear functional on $\mathfrak{A}_0$, which is normalised $\omega(\mathbb{1}) = 1$ and positive: $\omega(A^\dagger A) > 0$ for all $0 \neq A \in \mathfrak{A}_0$. We consider quasi-free states with vanishing one-point function, which are the analogue of the centred Gaussian covariance in Euclidean signature. That is, these states are characterised by the analogue of equation (17):

$$\omega(\phi(f_1) \star \cdots \star \phi(f_n)) = \sum_\pi \prod_{(i,j) \in \pi} \mathrm{i}\big(f_i, G^+ * f_j\big) \,, \tag{107}$$

where the sum runs over all partitions π of the set $\{1, \ldots, n\}$ into unordered pairs (i, j), and where

$$G^+(x, y) \equiv -\mathrm{i}\omega(\phi(x) \star \phi(y)) \tag{108}$$

is the two-point function of the state ω, here written in terms of its integral kernel. Summing, we also obtain the analogue of equation (18) for exponentials:

$$\omega\Big(\mathrm{e}_\star^{\mathrm{i}(J,\phi)}\Big) = \omega\left(\sum_{k=0}^\infty \frac{\mathrm{i}^k}{k!} \underbrace{(J, \phi) \star \cdots \star (J, \phi)}_{k \text{ times}} \right) = \mathrm{e}^{-\frac{\mathrm{i}}{2}(J, G^+ * J)} \,, \tag{109}$$

from which the above follows by functional differentiation with respect to J. Taking the expectation value of the commutation relations (105), we obtain

$$G^+(x, y) - G^+(y, x) = \mathit{\Delta}(x, y) \,, \tag{110}$$

such that the antisymmetric part of the two-point function is fixed.

For a massless scalar field in two-dimensional Minkowski space, we again have an IR divergence if we try to define the vacuum state as the limit $m \to 0$

of the massive one analogous to equation (24) in Euclidean signature. Namely, the massive two-point function reads

$$\begin{aligned} G^+(x,0) &= -2\pi\mathrm{i} \int \mathrm{e}^{\mathrm{i}px} \delta(p^2+m^2) \Theta(p^0) \frac{\mathrm{d}^2 p}{(2\pi)^2} \\ &= -\mathrm{i} \lim_{\epsilon\to 0^+} \int \frac{\mathrm{e}^{-\mathrm{i}\sqrt{(p^1)^2+m^2}x^0 + \mathrm{i}p^1x^1 - \epsilon\lvert p^1\rvert}}{2\sqrt{(p^1)^2+m^2}} \frac{\mathrm{d}p^1}{2\pi} \,, \end{aligned} \tag{111}$$

and if $m = 0$ the p^1 integral has a logarithmic singularity at the origin. In the limit $m \to 0$, we obtain

$$G^+(x,0) = \frac{\mathrm{i}}{4\pi} \ln\left[\frac{m^2 \mathrm{e}^{2\gamma}}{4} (\epsilon - \mathrm{i}(x^1-x^0))(\epsilon + \mathrm{i}(x^1+x^0)) \right] + \mathcal{O}(m) \,, \tag{112}$$

and we see that for spacelike separations $(x^1)^2 > (x^0)^2$ where we can set $\epsilon = 0$, we recover exactly the result (24) in Euclidean signature, up to an overall factor of $-\mathrm{i}$. However, the antisymmetric part has the well-defined limit

$$\begin{aligned} \Delta(x,0) &= G^+(x,0) - G^+(0,x) \\ &= \frac{\mathrm{i}}{4\pi} \lim_{\epsilon\to 0^+} \Big[\ln\big[\epsilon - \mathrm{i}(x^1-x^0)\big] + \ln\big[\epsilon + \mathrm{i}(x^1+x^0)\big] \\ &\qquad - \ln\big[\epsilon + \mathrm{i}(x^1-x^0)\big] - \ln\big[\epsilon - \mathrm{i}(x^1+x^0)\big] \Big] \\ &= \frac{1}{4} \big[\operatorname{sgn}(x^1-x^0) - \operatorname{sgn}(x^1+x^0) \big] = -\frac{1}{2} \Theta\big[(x^0)^2 - (x^1)^2\big] \operatorname{sgn}(x^0) \,, \end{aligned} \tag{113}$$

and vanishes outside the light cone as required. Because of the IR divergence, this state is not positive, and to construct the Hilbert space of the theory from the algebra $\mathfrak{A}_0$ and a state (via the GNS construction) the IR divergence must be cured, which can be done in different (related) ways:

- Working with a massive scalar field, and taking the limit $m \to 0$ only for expectation values of operators with a well-defined limit. This maintains positivity, and it is expected (and in some cases proven) that a finite mass arises from non-perturbative effects (*Debye screening*) [20, 40, 41].
- The separation of the constant part of ϕ and its quantisation as a massless harmonic oscillator, similar to what is done in string theory [53] and de Sitter QFT [54]. In pAQFT, this is the Dereziński–Meissner representation [55].
- A Krein space construction, where positivity is only maintained in the physical subspace which contains derivatives of ϕ and vertex operators [56, 57].
- Restricting the algebra $\mathfrak{A}_0$ to be generated by derivatives of ϕ and vertex operators, which we will do in the following.

All three constructions are equivalent for our purposes, since the interacting expectation values of $\mathcal{O}_{\mu\nu}$ and $T_{\mu\nu}$, defined again using the Gell-Mann–Low formula (10) only involves derivatives of ϕ and vertex operators, which have a well-defined massless limit.

Instead of the vacuum state, we take moreover a general quasi-free Hadamard state, which for our purposes can be defined as the quasi-free state $\omega^{\Lambda,\epsilon}$ with two-point function

$$G^+(x,y) = \frac{\mathrm{i}}{4\pi} \ln\left[\Lambda^2 (\epsilon + \mathrm{i}u)(\epsilon + \mathrm{i}v) \right] - \mathrm{i}W(x,y) \,, \tag{114}$$

where $W(x,y)$ is a smooth and symmetric bisolution of the massless Klein–Gordon equation $\partial_x^2 W(x,y) = \partial_y^2 W(x,y) = 0$, and we introduced the light cone coordinates

$$u = u(x,y) \equiv (x^0 - y^0) - (x^1 - y^1) \,, \quad v = v(x,y) \equiv (x^0 - y^0) + (x^1 - y^1) \,. \tag{115}$$

As in the Euclidean case, Λ is an IR cutoff which we ultimately take to vanish, and we keep $\epsilon > 0$ as a UV cutoff. That is, the physical two-point function is obtained as the distributional boundary value (in the limit $\epsilon \to 0$) from the function (114) which is analytic for all $\epsilon > 0$. The two-point function (114) can be decomposed as

$$G^+(x,y) = H^+(x,y) + \frac{\mathrm{i}}{2\pi} \ln\left(\frac{\Lambda}{\mu} \right) - \mathrm{i}W(x,y) \,, \tag{116}$$

where

$$H^+(x,y) \equiv \frac{\mathrm{i}}{4\pi} \ln\left[\mu^2 (\epsilon + \mathrm{i}u)(\epsilon + \mathrm{i}v) \right] \tag{117}$$

is the Hadamard parametrix containing the singular part of the two-point function, which is the same for all Hadamard states. The Feynman propagator and parametrix are the time-ordered versions of (114) and (117), and read

$$\begin{aligned} G^\mathrm{F}(x,y) &\equiv \Theta(x^0 - y^0) G^+(x,y) + \Theta(y^0 - x^0) G^+(y,x) \\ &= H^\mathrm{F}(x,y) + \frac{\mathrm{i}}{2\pi} \ln\left(\frac{\Lambda}{\mu} \right) - \mathrm{i}W(x,y) \,, \end{aligned} \tag{118a}$$

$$\begin{aligned} H^\mathrm{F}(x,y) &\equiv \Theta(x^0 - y^0) H^+(x,y) + \Theta(y^0 - x^0) H^+(y,x) \\ &= \frac{\mathrm{i}}{4\pi} \ln\left[\mu^2 (-uv + \mathrm{i}\epsilon \lvert u + v \rvert + \epsilon^2) \right] \,, \end{aligned} \tag{118b}$$

where we used that $W(x,y)$ is symmetric. We note that the time-ordered Hadamard parametrix H^F is a fundamental solution of the massless Klein–Gordon equation:

$$\begin{aligned} \partial^2 H^\mathrm{F}(x,y) = -4 \partial_u \partial_v H^\mathrm{F}(x,y) &= \frac{2\epsilon}{\pi (u^2 + \epsilon^2)} \delta(u+v) \\ &\to 2\delta(u)\delta(v) = \delta^2(x-y) \quad (\epsilon \to 0) \,, \end{aligned} \tag{119}$$

where the second equality is a straightforward computation using the well-known results $\lvert x \rvert' = \operatorname{sgn}(x)$, $\Theta'(x) = \delta(x)$ and $\operatorname{sgn}'(x) = 2\delta(x)$, and the limit

$$\lim_{\epsilon \to 0} \frac{\epsilon}{x^2 + \epsilon^2} = \mathfrak{Im} \lim_{\epsilon \to 0} \frac{1}{x - \mathrm{i}\epsilon} = \pi \delta(x) \,. \tag{120}$$

is the Sokhotski–Plemelj theorem.

The algebra $\mathfrak{A}_0$ is completed by adding normal-ordered products $\mathcal{N}$. Normal ordering can be performed with respect to the full two-point function G^+

or the Hadamard parametrix H^+; the latter choice has the advantage that the normal-ordered products transform covariantly under diffeomorphisms, or Lorentz transformations in Minkowski space. Analogously to equation (25), we have for exponentials

$$\mathcal{N}_G\left[\mathrm{e}^{\mathrm{i}(J,\phi)}\right] = \mathrm{e}^{\frac{\mathrm{i}}{2}(J,G^+ * J)}\mathrm{e}_\star^{\mathrm{i}(J,\phi)} \tag{121}$$

with $J \in \mathcal{S}(\mathbb{R}^2)$ (and the analogous formula with G replaced by H), from which the normal-ordering of monomials is obtained by functional differentiation with respect to J, assuming that the right-hand side is a well-defined element of $\mathfrak{A}_0$. In the massless case, that means that we have to take $\int J(x)\,\mathrm{d}^2x = 0$, which ensures that it can be written as $J(x) = \partial_\mu J^\mu(x)$ (see, e.g., [58, App. B and C]) such that $(J,\phi) = -(J^\mu, \partial_\mu\phi)$ and only derivatives of ϕ enter. The expectation value of a normal-ordered quantity is given by

$$\omega^{\Lambda,\epsilon}(\mathcal{N}_G[(J_1,\phi)\cdots(J_n,\phi)]) = \delta_{n,0} \tag{122}$$

analogous to equation (27), and for a change in normal ordering we have

$$\mathcal{N}_G\left[\mathrm{e}^{\mathrm{i}(J,\phi)}\right] = \mathrm{e}^{\frac{\mathrm{i}}{2}(J,(G^+-H^+)*J)}\,\mathcal{N}_H\left[\mathrm{e}^{\mathrm{i}(J,\phi)}\right], \tag{123}$$

both formulas with the above restriction on J. The vertex operators $V_\alpha(x)$ are formally given by the exponentials $\mathrm{e}^{\mathrm{i}\alpha\phi(x)}$, but those cannot be defined if we only consider derivatives of ϕ. Instead, we include the Hadamard-normal-ordered operators $\mathcal{N}_H[V_\alpha(x)]$ among the generators of the algebra $\mathfrak{A}_0$, with a change of normal ordering analogous to equation (29) given by

$$\mathcal{N}_G[V_\alpha(x)] = \mathrm{e}^{\frac{\mathrm{i}}{2}\alpha^2(G^+-H^+)(x,x)}\,\mathcal{N}_H[V_\alpha(x)] = \left(\frac{\Lambda}{\mu}\right)^{-\frac{\alpha^2}{4\pi}}\mathrm{e}^{\frac{\alpha^2}{2}W(x,x)}\,\mathcal{N}_H[V_\alpha(x)] \tag{124}$$

using the decomposition of the two-point function (116), and expectation value

$$\omega^{\Lambda,\epsilon}(\mathcal{N}_G[V_\alpha(x)]) = 1\,. \tag{125}$$

It remains to define their $\star$ products with other elements of $\mathfrak{A}_0$. For this, we note that equation (121) implies

$$\begin{aligned}
\mathcal{N}_G\left[\mathrm{e}^{\mathrm{i}(J,\phi)}\right] \star \mathcal{N}_G\left[\mathrm{e}^{\mathrm{i}(K,\phi)}\right] &= \mathrm{e}^{\frac{\mathrm{i}}{2}(J,G^+*J)+\frac{\mathrm{i}}{2}(K,G^+*K)}\mathrm{e}_\star^{\mathrm{i}(J,\phi)} \star \mathrm{e}_\star^{\mathrm{i}(K,\phi)} \\
&= \exp\left[\frac{\mathrm{i}}{2}\Big((J,G^+ * J) + (K,G^+ * K) - (J,\Delta * K)\Big)\right]\mathrm{e}_\star^{\mathrm{i}(J+K,\phi)} \\
&= \exp\big[-\mathrm{i}(J,G^+ * K)\big]\,\mathcal{N}_G\left[\mathrm{e}^{\mathrm{i}(J+K,\phi)}\right]
\end{aligned} \tag{126}$$

using the Baker–Campbell–Hausdorff formula [59] and the commutation relations (105) and (110), as well as the analogous formula with G replaced by H. Taking functional derivatives with respect to J or K and taking into account the condition $\int J(x)\,\mathrm{d}^2x = 0$, we obtain the product for terms involving powers of $\partial_\mu\phi$, while setting $J(x) = \alpha\delta^2(x-y)$ and interpreting $\mathrm{e}^{\mathrm{i}\alpha\phi(y)}$ as vertex operator $V_\alpha(y)$, we obtain the products for them. It follows that $\mathfrak{A}_0$ also contains terms of the form $\mathcal{N}_H[V_{\alpha_1}(x_1) \cdots V_{\alpha_n}(x_n) \partial_{\mu_1}\phi(z_1) \cdots \partial_{\mu_k}\phi(z_k)]$ and similar, i.e., normal-ordered products of multiple vertex operators and derivatives of ϕ. Generalising equation (125), their expectation value (after changing the normal ordering to involve G) vanishes if $k > 0$, and is equal to 1 if $k = 0$ for any n.

A central object in pAQFT are time-ordered products $\mathcal{T}$, which can be defined as multilinear maps from classical expressions into $\mathfrak{A}_0$. They are constructed inductively, using that the ones with single entries are equal to the Hadamard-normal-ordered products

$$\mathcal{T}[\mathcal{O}(x)] = \mathcal{N}_H[\mathcal{O}(x)]\,, \tag{127}$$

while the higher ones are defined outside the diagonal by causal factorisation:

$$\begin{aligned} &\mathcal{T}[\mathcal{O}_1(x_1) \otimes \cdots \otimes \mathcal{O}_n(x_n)] \\ &\quad = \mathcal{T}[\mathcal{O}_1(x_1) \otimes \cdots \otimes \mathcal{O}_k(x_k)] \star \mathcal{T}[\mathcal{O}_{k+1}(x_{k+1}) \otimes \cdots \otimes \mathcal{O}_n(x_n)] \end{aligned} \tag{128}$$

if none of the $x_1, \ldots, x_k$ lie in the past light cone of any of the $x_{k+1}, \ldots, x_n$. The extension to the total diagonal $x_1 = \cdots = x_n$ corresponds to renormalisation. We then want to prove the analogue of Lemma 1 (for $n = 2$), which is

Lemma 2. *The expectation value of a time-ordered product of vertex operators and a bilinear operator is given by*

$$\begin{aligned} \omega^{\Lambda,\epsilon}\left(\mathcal{T}\left[\bigotimes_{j=1}^n V_{\alpha_j}(x_j) \otimes (\partial_{\vec{\mu}}\phi\, \partial_{\vec{\nu}}\phi)(z) \right] \right) &= \exp\left[-\mathrm{i} \sum_{1 \leq i < j \leq n} \alpha_i \alpha_j H^F(x_i, x_j) \right] \\ &\times \left[\lim_{z' \to z} \partial^z_{\vec{\mu}} \partial^{z'}_{\vec{\nu}} W(z, z') + \sum_{i,j=1}^n \alpha_i \alpha_j \partial_{\vec{\mu}} G^F(z, x_i) \partial_{\vec{\nu}} G^F(z, x_j) \right] \\ &\times \exp\left[-\frac{1}{2} \sum_{i,j=1}^n \alpha_i \alpha_j W(x_i, x_j) \right] \left(\frac{\Lambda}{\mu} \right)^{\frac{\left(\sum_{k=1}^n \alpha_k\right)^2}{4\pi}}\,, \end{aligned} \tag{129}$$

where $\partial_{\vec{\mu}} \equiv \partial_{\mu_1} \cdots \partial_{\mu_k}$ and $\partial_{\vec{\nu}} \equiv \partial_{\nu_1} \cdots \partial_{\nu_\ell}$ with $k, \ell \geq 1$.

Proof. The result (129) follows immediately from the expression

$$
\begin{aligned}
\mathcal{T}\left[\bigotimes_{j=1}^{n} V_{\alpha_j}(x_j) \otimes (\partial_{\vec{\mu}}\phi\, \partial_{\vec{\nu}}\phi)(z)\right] &= \exp\left[-\mathrm{i} \sum_{1\leq i<j\leq n} \alpha_i\alpha_j H^{\mathrm{F}}(x_i,x_j)\right] \\
&\times \left[\sum_{i,j=1}^{n} \alpha_i\alpha_j \partial_{\vec{\mu}} G_{\mathrm{F}}(z,x_i)\partial_{\vec{\nu}} G_{\mathrm{F}}(z,x_j)\,\mathcal{N}_G\left[\prod_{j=1}^{n} V_{\alpha_j}(x_j)\right]\right. \\
&\quad - 2\sum_{i=1}^{n} \alpha_i \partial_{(\vec{\mu}} G_{\mathrm{F}}(z,x_i)\,\mathcal{N}_G\left[\partial_{\vec{\nu})}\phi(z)\prod_{j=1}^{n} V_{\alpha_j}(x_j)\right] \\
&\quad \left. + \mathcal{N}_G\left[(\partial_{\vec{\mu}}\phi\,\partial_{\vec{\nu}}\phi)(z)\prod_{j=1}^{n} V_{\alpha_j}(x_j)\right] + \lim_{z'\to z} \partial^{z}_{\vec{\mu}}\partial^{z'}_{\vec{\nu}} W(z,z')\,\mathcal{N}_G\left[\prod_{j=1}^{n} V_{\alpha_j}(x_j)\right]\right] \\
&\times \exp\left[-\frac{1}{2}\sum_{i,j=1}^{n} \alpha_i\alpha_j W(x_i,x_j)\right] \left(\frac{\Lambda}{\mu}\right)^{\frac{\left(\sum_{k=1}^{n}\alpha_k\right)^2}{4\pi}}
\end{aligned}
\tag{130}
$$

for the time-ordered product, using that the expectation value of a normal-ordered expression (with respect to G) involving powers of ϕ vanishes, while it is equal to 1 if only vertex operators appear. To prove equation (130), we first have to show the related result

$$
\begin{aligned}
\mathcal{T}\left[\bigotimes_{j=1}^{n} V_{\alpha_j}(x_j)\right] = \exp\left[-\mathrm{i}\sum_{1\leq i<j\leq n} \alpha_i\alpha_j H^{\mathrm{F}}(x_i,x_j)\right] \mathcal{N}_G\left[\prod_{j=1}^{n} V_{\alpha_j}(x_j)\right] \\
\times \exp\left[-\frac{1}{2}\sum_{i,j=1}^{n} \alpha_i\alpha_j W(x_i,x_j)\right] \left(\frac{\Lambda}{\mu}\right)^{\frac{\left(\sum_{k=1}^{n}\alpha_k\right)^2}{4\pi}}
\end{aligned}
\tag{131}
$$

by induction in n. Note that both sides are symmetric under a permutation of the α_j and x_j, the right-hand side because the Feynman Hadamard parametrix H^{F} (118) is symmetric in its arguments. For $n = 1$, we compute

$$
\mathcal{T}[V_\alpha(x)] = \mathcal{N}_H[V_\alpha(x)] = \left(\frac{\Lambda}{\mu}\right)^{\frac{\alpha^2}{4\pi}} \mathrm{e}^{-\frac{\alpha^2}{2}W(x,x)}\,\mathcal{N}_G[V_\alpha(x)] \tag{132}
$$

using equation (127) and the change of normal-ordering for vertex operators (124), which is the correct result. Assume thus that equation (131) is fulfilled for all $m \leq n$, and consider the time-ordered products with $n+1$ vertex operators. If not all points coincide, k of them are not in the past light cone of any of the other $n+1-k$ for some $1 \leq k \leq n$, and by relabeling we may assume that these are the first. Using the causal factorisation (128), we thus obtain

$$\begin{aligned}
\mathcal{T}\left[\bigotimes_{j=1}^{n+1} V_{\alpha_j}(x_j)\right] &= \mathcal{T}\left[\bigotimes_{j=1}^{k} V_{\alpha_j}(x_j)\right] \star \mathcal{T}\left[\bigotimes_{j=k+1}^{n+1} V_{\alpha_j}(x_j)\right] \\
&= \exp\left[-\mathrm{i} \sum_{1\leq i<j\leq k} \alpha_i\alpha_j H^{\mathrm{F}}(x_i,x_j) - \mathrm{i} \sum_{k+1\leq i<j\leq n+1} \alpha_i\alpha_j H^{\mathrm{F}}(x_i,x_j)\right] \\
&\quad \times \mathcal{N}_G\left[\prod_{i=1}^{k} V_{\alpha_i}(x_i)\right] \star \mathcal{N}_G\left[\prod_{j=k+1}^{n+1} V_{\alpha_j}(x_j)\right] \left(\frac{\Lambda}{\mu}\right)^{\frac{\left(\sum_{i=1}^{k}\alpha_i\right)^2 + \left(\sum_{j=k+1}^{n+1}\alpha_j\right)^2}{4\pi}} \\
&\quad \times \exp\left[-\frac{1}{2}\sum_{i,j=1}^{k} \alpha_i\alpha_j W(x_i,x_j) - \frac{1}{2}\sum_{i,j=k+1}^{n+1} \alpha_i\alpha_j W(x_i,x_j)\right] ,
\end{aligned} \tag{133}$$

using the induction hypothesis in the second equality. Employing equation (126) for the star product of two normal-ordered expressions (with the exponentials interpreted as vertex operators as explained there) and the decomposition of the two-point function (116), we obtain

$$\begin{aligned}
&\mathcal{N}_G\left[\prod_{i=1}^{k} V_{\alpha_i}(x_i)\right] \star \mathcal{N}_G\left[\prod_{j=k+1}^{n+1} V_{\alpha_j}(x_j)\right] \\
&\quad = \exp\left[-\mathrm{i}\sum_{i=1}^{k}\sum_{j=k+1}^{n+1} \alpha_i\alpha_j G^{+}(x_i,x_j)\right] \mathcal{N}_G\left[\prod_{j=1}^{n+1} V_{\alpha_j}(x_j)\right] \\
&\quad = \exp\left[-\mathrm{i}\sum_{i=1}^{k}\sum_{j=k+1}^{n+1} \alpha_i\alpha_j H^{+}(x_i,x_j)\right] \mathcal{N}_G\left[\prod_{j=1}^{n+1} V_{\alpha_j}(x_j)\right] \\
&\qquad \times \exp\left[-\sum_{i=1}^{k}\sum_{j=k+1}^{n+1} \alpha_i\alpha_j W(x_i,x_j)\right] \left(\frac{\Lambda}{\mu}\right)^{\frac{\sum_{i=1}^{k}\sum_{j=k+1}^{n+1}\alpha_i\alpha_j}{2\pi}} .
\end{aligned} \tag{134}$$

Using that $H^{+}(x_i,x_j) = H^{\mathrm{F}}(x_i,x_j)$ if x_i does not lie in the past light cone of x_j as we have assumed, and inserting the result (134) into equation (133), one easily sees that the various terms combine into the required form (131), which therefore holds at least outside the diagonal. To extend the result to the diagonal, we note that since the right-hand side is a smooth function of the x_i if $\epsilon > 0$, it simply extends by continuity. This is even true in the limit of vanishing UV cutoff ϵ if $\alpha_i^2 < 4\pi$ for all i, since then the scaling degree of $\exp\left[-\mathrm{i}\sum_{1\leq i<j\leq n+1} \alpha_i\alpha_j H^{\mathrm{F}}(x_i,x_j)\right]$ is less than $2(m-1) = (m-1)\dim\mathbb{R}^2$ on each subdiagonal where m points coincide, such that the singularities that arise there are integrable.

We can now prove equation (130), which we also do by induction in n. For $n = 0$, we compute

$$\begin{aligned}\mathcal{T}[(\partial_{\vec{\mu}}\phi\,\partial_{\vec{\nu}}\phi)(z)] &= \mathcal{N}_H[(\partial_{\vec{\mu}}\phi\,\partial_{\vec{\nu}}\phi)(z)] \\ &= \mathcal{N}_G[(\partial_{\vec{\mu}}\phi\,\partial_{\vec{\nu}}\phi)(z)] + \lim_{z'\to z} \partial^z_{\vec{\mu}}\partial^{z'}_{\vec{\nu}} W(z,z')\end{aligned} \tag{135}$$

using equation (127), the change of normal-ordering (123) at second order in J [taking once $J(x) = \partial_{\vec{\mu}}\delta^2(z-x)$ and once $J(x) = \partial_{\vec{\nu}}\delta^2(z-x)$], as well as the decomposition of the two-point function (116). Since this agrees with equation (130) for $n = 0$, the base case is proven. Assume thus that equation (130) is fulfilled for all $m \leq n$, and consider the time-ordered products with $n+1$ vertex operators. If not all points coincide, again k of them will not lie in the past light cone of the other $n+2-k$ for some $1 \leq k \leq n+1$. We thus can again use causal factorisation (128), but now have to distinguish two cases, namely whether the distinguished point z is among the group of k or among the group of the $n+2-k$ points. In the first case, we have

$$\begin{aligned}&\mathcal{T}\left[\bigotimes_{j=1}^{n+1} V_{\alpha_j}(x_j) \otimes (\partial_{\vec{\mu}}\phi\,\partial_{\vec{\nu}}\phi)(z)\right] \\ &\quad= \mathcal{T}\left[\bigotimes_{j=1}^{k-1} V_{\alpha_j}(x_j) \otimes (\partial_{\vec{\mu}}\phi\,\partial_{\vec{\nu}}\phi)(z)\right] \star \mathcal{T}\left[\bigotimes_{j=k}^{n+1} V_{\alpha_j}(x_j)\right],\end{aligned} \tag{136}$$

and inserting the induction hypothesis and the previous result (131) on the right-hand side, we obtain a number of star products of normal-ordered expressions, which are too long to display explicitly. To evaluate them, we need in addition to equation (134) also

$$\begin{aligned}&\mathcal{N}_G\left[\partial_{\vec{\mu}}\phi(z)\prod_{j=1}^{k-1} V_{\alpha_j}(x_j)\right] \star \mathcal{N}_G\left[\prod_{j=k}^{n+1} V_{\alpha_j}(x_j)\right] \\ &\quad= \exp\left[-\mathrm{i}\sum_{i=1}^{k-1}\sum_{j=k}^{n+1} \alpha_i\alpha_j G^+(x_i,x_j)\right] \mathcal{N}_G\left[\partial_{\vec{\mu}}\phi(z)\prod_{j=1}^{n+1} V_{\alpha_j}(x_j)\right] \\ &\qquad - \sum_{j=k}^{n+1} \alpha_j \partial_{\vec{\mu}} G^+(z,x_j) \exp\left[-\mathrm{i}\sum_{i=1}^{k-1}\sum_{j=k}^{n+1} \alpha_i\alpha_j G^+(x_i,x_j)\right] \mathcal{N}_G\left[\prod_{j=1}^{n+1} V_{\alpha_j}(x_j)\right]\end{aligned} \tag{137}$$

and

$$
\begin{aligned}
&\mathcal{N}_G \left[(\partial_{\vec{\mu}} \phi \, \partial_{\vec{\nu}} \phi)(z) \prod_{j=1}^{k-1} V_{\alpha_j}(x_j) \right] \star \mathcal{N}_G \left[\prod_{j=k}^{n+1} V_{\alpha_j}(x_j) \right] \\
&\quad = \exp\left[-\mathrm{i} \sum_{i=1}^{k-1} \sum_{j=k}^{n+1} \alpha_i \alpha_j G^+(x_i, x_j) \right] \Bigg[\mathcal{N}_G \left[(\partial_{\vec{\mu}} \phi \, \partial_{\vec{\nu}} \phi)(z) \prod_{j=1}^{n+1} V_{\alpha_j}(x_j) \right] \\
&\qquad - 2 \sum_{j=k}^{n+1} \alpha_j \partial_{(\vec{\mu}} G^+(z, x_j) \, \mathcal{N}_G \left[\partial_{\vec{\nu})} \phi(z) \prod_{j=1}^{n+1} V_{\alpha_j}(x_j) \right] \\
&\qquad + \sum_{i,j=k}^{n+1} \alpha_i \alpha_j \partial_{\vec{\mu}} G^+(z, x_i) \partial_{\vec{\nu}} G^+(z, x_j) \, \mathcal{N}_G \left[\prod_{j=1}^{n+1} V_{\alpha_j}(x_j) \right] \Bigg] ,
\end{aligned}
\tag{138}
$$

which are obtained by first taking one or two functional derivatives of equation (126) with respect to J, and then interpreting the exponentials as vertex operators, setting $J(x) = \sum_{j=1}^{k-1} \alpha_j \delta^2(x - x_j)$ and $K(x) = \sum_{j=k}^{n+1} \alpha_j \delta^2(x - x_j)$. Using further the decomposition of the two-point function (116) and the fact that for x_i not in the past light cone of x_j we have $H^+(x_i, x_j) = H^{\mathrm{F}}(x_i, x_j)$, the result (130) follows in this case.

A similar computation yields equation (130) also in the second case where the distinguished point z is among the second group of $n + 2 - k$ points, for which we need the analogues of equations (137) and (138) with the factors reversed. We omit the details. We have thus shown that equation (130) holds at least outside the diagonal, but since the right-hand side is a smooth function of the x_i if $\epsilon > 0$, it extends to the diagonal by continuity. However, this is no longer true for $\epsilon = 0$, and we resolve the renormalisation problem in section 3.2. □

Remark. While the choice of taking the Hadamard-normal-ordered expressions for time-ordered products with a single entry (127) may seem strange to someone aquainted with flat-space quantum field theory, it is actually indispensable in curved spacetimes, since otherwise the renormalisation freedom is unacceptably large [60]. Moreover, in flat space of three or more dimensions the Hadamard parametrix actually coincides with the vacuum two-point function. It is only in two dimensions or for non-vacuum states that the difference becomes relevant, and as we see from equation (129) it is the correct choice to ensure the superselection rule in analogy with the Euclidean case.

3.2. Proof of theorem 4 (Renormalisation). We begin again with $\mathcal{O}_{\mu\nu} = \partial_\mu \phi \, \partial_\nu \phi$. Using Lemma 2, the decomposition of the two-point function and the explicit form of the Hadamard parametrix (118), we obtain (with $\sigma_j = \pm 1$)

$$\begin{aligned}
\omega^{\Lambda,\epsilon}\left(\mathcal{T}\left[\bigotimes_{j=1}^{n} V_{\sigma_j\beta}(x_j)\otimes\mathcal{O}_{\mu\nu}(z)\right]\right) &= \prod_{1\le i<j\le n}\left[\mu^2(-u_{ij}v_{ij}+\mathrm{i}\epsilon)\right]^{\sigma_i\sigma_j\frac{\beta^2}{4\pi}} \\
\times\Bigg[\lim_{z'\to z}\partial^z_\mu\partial^{z'}_\nu W(z,z') &- \beta^2\sum_{i,j=1}^{n}\sigma_i\sigma_j\partial_\mu W(z,x_i)\partial_\nu W(z,x_j) \\
&+\frac{\beta^2}{2\pi}\sum_{i,j=1}^{n}\sigma_i\sigma_j H_{(\mu}(z,x_i)\partial_{\nu)}W(z,x_j) \\
&-\frac{\beta^2}{(4\pi)^2}\sum_{i,j=1}^{n}\sigma_i\sigma_j H_\mu(z,x_i)H_\nu(z,x_j)\Bigg] \\
\times\exp\left[-\frac{\beta^2}{2}\sum_{i,j=1}^{n}\sigma_i\sigma_j W(x_i,x_j)\right]&\left(\frac{\Lambda}{\mu}\right)^{\frac{\beta^2\left(\sum_{k=1}^{n}\sigma_k\right)^2}{4\pi}},
\end{aligned} \tag{139}$$

where we defined

$$H_\mu(x,y)\equiv -4\pi\mathrm{i}\,\partial_\mu H^{\mathrm{F}}(x,y) = \frac{\Theta(u+v)}{x^\mu-\mathrm{i}\epsilon}+\frac{\Theta(-(u+v))}{x^\mu+\mathrm{i}\epsilon}\,. \tag{140}$$

and set (for better readability)

$$u_{ij}\equiv u(x_i,x_j)\,,\quad v_{ij}\equiv v(x_i,x_j)\,. \tag{141}$$

Analogously to the Euclidean case, the terms $\left[\mu^2(-u^{ij}v^{ij}+\mathrm{i}\epsilon)\right]^{\sigma_j\sigma_k\frac{\beta^2}{4\pi}}$ are singular in the physical limit $\epsilon\to 0$ if $\sigma_j\sigma_k = -1$, but the singularity is integrable since we are in the finite regime $\beta^2 < 4\pi$. Since W is a smooth function, it is not singular, but since the scaling degree of H_μ (140) is -1 and the integration measure in light cone coordinates (115) is $\mathrm{d}^2x = \frac{1}{2}\,\mathrm{d}u\,\mathrm{d}v$, terms involving H_μ can potentially be problematic and may need renormalisation. The terms of the form $H_{(\mu}(z,x_i)\partial_{\nu)}W(z,x_j)$ are by definition (140) equal to $-4\pi\mathrm{i}\,\partial^z_{(\mu}H^{\mathrm{F}}(z,x_i)\partial_{\nu)}W(z,x_j)$, and we can integrate the z derivative by parts such that it either acts on W or the smearing function. Since the singularity of H^{F} is integrable, the result is a well-defined distribution and we can take the limit $\epsilon\to 0$ with impunity. The same holds for the terms $H_\mu(z,x_i)H_\nu(z,x_j)$ with $i\neq j$ since their scaling degree (when all points coincide) is $2<\dim\mathbb{R}^4$, but as in the Euclidean case the terms $H_\mu(z,x_j)H_\nu(z,x_j)$ are problematic since their scaling degree is $2=\dim\mathbb{R}^2$ such that we expect a logarithmic singularity as $\epsilon\to 0$. To determine the required counterterm, we compute first

$$\begin{aligned}
H_u(x,y)H_u(x,y) &= \frac{\Theta(u+v)}{(u-\mathrm{i}\epsilon)^2}+\frac{\Theta(-(u+v))}{(u+\mathrm{i}\epsilon)^2} \\
&= -\partial_u H_u(x,y)+\frac{2\mathrm{i}\epsilon}{u^2+\epsilon^2}\delta(u+v) \\
&\to 4\pi\mathrm{i}\,\partial_u^2 H^{\mathrm{F}}(x,y)+\mathrm{i}\pi\delta^2(x-y)\quad(\epsilon\to 0)\,,
\end{aligned} \tag{142}$$

where we used the Sokhotski–Plemelj formula (120) for the last limit. Integrating the u derivatives by parts on the test function, the singularity of H^F is integrable and we have a well-defined distribution. In the same way, we obtain

$$H_v(x,y) H_v(x,y) \to 4\pi\mathrm{i}\, \partial_v^2 H^\mathrm{F}(x,y) + \mathrm{i}\pi \delta^2(x-y) \quad (\epsilon \to 0)\,, \tag{143}$$

but the mixed terms are more complicated. For them, we compute

$$\begin{aligned} H_u(x,y) H_v(x,y) &= \frac{\Theta(u+v)}{(u-\mathrm{i}\epsilon)(v-\mathrm{i}\epsilon)} + \frac{\Theta(-(u+v))}{(u+\mathrm{i}\epsilon)(v+\mathrm{i}\epsilon)} \\ &= -8\pi^2 \partial_u \partial_v \left[H^\mathrm{F}(x,y) \right]^2 - \frac{2\mathrm{i}\epsilon \ln\left[\mu^2 \left(u^2 + \epsilon^2 \right) \right]}{u^2+\epsilon^2} \delta(u+v) \end{aligned} \tag{144}$$

using that H^F is a fundamental solution (119), and the limit of the last term is again most easily computed in Fourier space: we have for $\Re\mathfrak{e}\, \alpha > 0$ [46, Eq. 10.32.11]

$$\int \frac{\mathrm{e}^{-\mathrm{i}pu}}{(u^2+\epsilon^2)^\alpha} \mathop{}\!\mathrm{d} u = \frac{2\sqrt{\pi} \lvert p \rvert^{\alpha-\frac{1}{2}}}{\Gamma(\alpha) (2\epsilon)^{\alpha-\frac{1}{2}}} \mathrm{K}_{\alpha-\frac{1}{2}}(\lvert p \rvert \epsilon)\,, \tag{145}$$

where K is the second modified Bessel function. It follows that

$$\begin{aligned} \frac{\ln\left[\mu^2 \left(u^2 + \epsilon^2 \right) \right]}{u^2+\epsilon^2} &= \lim_{\delta \to 0} \left[\frac{1}{\delta} \left(\mu^{2\delta} (u^2+\epsilon^2)^{-1+\delta} - (u^2+\epsilon^2)^{-1} \right) \right] \\ &= \lim_{\delta \to 0} \int \left[\frac{1}{\delta} \left(\mu^{2\delta} \frac{2\sqrt{\pi} \lvert p \rvert^{\frac{1}{2}-\delta}}{\Gamma(1-\delta) (2\epsilon)^{\frac{1}{2}-\delta}} \mathrm{K}_{\frac{1}{2}-\delta}(\lvert p \rvert \epsilon) - \frac{2\sqrt{\pi} \lvert p \rvert^{\frac{1}{2}}}{(2\epsilon)^{\frac{1}{2}}} \mathrm{K}_{\frac{1}{2}}(\lvert p \rvert \epsilon) \right) \right] \mathrm{e}^{\mathrm{i}pu} \frac{\mathrm{d} p}{2\pi} \\ &= \frac{\pi}{\epsilon} \int \mathrm{e}^{-\lvert p \rvert \epsilon} \left[2 \ln(2\mu\epsilon) - \mathrm{e}^{2\lvert p \rvert \epsilon} \mathrm{E}_1(2\lvert p \rvert \epsilon) - \gamma - \ln(2\lvert p \rvert \epsilon) \right] \mathrm{e}^{\mathrm{i}pu} \frac{\mathrm{d} p}{2\pi}\,, \end{aligned} \tag{146}$$

where E_1 is the exponential integral and we used the value of the modified Bessel function K_ν and its first derivative with respect to ν at $\nu = \frac{1}{2}$ [46, Eqs. (10.39.2) and (10.38.7)]. Using the known asymptotic expansion of E_1 for small argument [46, Eq. (6.6.2)], we obtain in the limit $\epsilon \to 0$ that

$$\epsilon \frac{\ln\left[\mu^2 \left(u^2 + \epsilon^2 \right) \right]}{u^2+\epsilon^2} \approx 2\pi \ln(2\mu\epsilon) \int \mathrm{e}^{\mathrm{i}pu} \frac{\mathrm{d} p}{2\pi} = 2\pi \ln(2\mu\epsilon)\, \delta(u) \tag{147}$$

and thus

$$H_u(x,y) H_v(x,y) \to -8\pi^2 \partial_u \partial_v \left[H^\mathrm{F}(x,y) \right]^2 - 2\pi\mathrm{i} \ln(2\mu\epsilon)\, \delta^2(x-y)\,, \tag{148}$$

where the first term is a well-defined distribution (the renormalised part, i.e., the extension of $H_u(x,y) H_v(x,y)$ to the diagonal), and the second term is the anticipated logarithmically divergent local term as the UV cutoff ϵ is removed.

In light cone coordinates (115), the Minkowski metric reads

$$\mathrm{d} s^2 = - \mathrm{d} u \mathop{}\!\mathrm{d} v = \eta_{\mu\nu} \mathop{}\!\mathrm{d} x^\mu \mathop{}\!\mathrm{d} x^\nu \quad \Rightarrow \quad \eta_{\mu\nu} = - \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \eta^{\mu\nu} = -2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{149}$$

such that we can summarise the above results as

$$H_\mu(x,y)H_\nu(x,y) = [H_\mu(x,y)H_\nu(x,y)]^{\text{ren}} + 4\pi\mathrm{i}\,\eta_{\mu\nu}\ln(2\mu\epsilon)\,\delta^2(x-y) + \mathcal{O}(\epsilon)\,, \tag{150}$$

using that $\delta^2(x-y) = 2\delta(u)\delta(v)$ and with

$$[H_u(x,y)H_u(x,y)]^{\text{ren}} = 4\pi\mathrm{i}\,\partial_u^2 H^{\text{F}}(x,y) + \mathrm{i}\pi\delta^2(x-y)\,, \tag{151a}$$

$$[H_v(x,y)H_v(x,y)]^{\text{ren}} = 4\pi\mathrm{i}\,\partial_v^2 H^{\text{F}}(x,y) + \mathrm{i}\pi\delta^2(x-y)\,, \tag{151b}$$

$$[H_u(x,y)H_v(x,y)]^{\text{ren}} = -8\pi^2\partial_u\partial_v\left[H^{\text{F}}(x,y)\right]^2. \tag{151c}$$

To renormalise, we have to subtract the local term, which in the pAQFT framework is done by changing the time-ordered products by local terms. In our case, with the local term supported at the diagonal $z = x_i$, we thus have to change the time-ordered products with two entries, one of which is $\mathcal{O}_{\mu\nu}(z)$ and one of which is a vertex operator $V_{\sigma_i\beta}(x_i)$. Since the local term that we want to subtract has scaling dimension 2, the same as $\mathcal{O}_{\mu\nu}$, the time-ordered product must be proportional to the vertex operator, and we make the ansatz

$$\delta\mathcal{T}[\mathcal{O}_{\mu\nu}(z)\otimes V_\alpha(x)] = c_\alpha\,\eta_{\mu\nu}\delta^2(z-x)\,\mathcal{T}[V_\alpha(x)] \tag{152}$$

with a constant c_α to be determined. From the recursive construction of the time-ordered products, it follows that

$$\delta\mathcal{T}\left[\bigotimes_{j=1}^n V_{\sigma_j\beta}(x_j)\otimes\mathcal{O}_{\mu\nu}(z)\right] = c_{\sigma_j\beta}\,\eta_{\mu\nu}\sum_{j=1}^n\delta^2(z-x_j)\,\mathcal{T}\left[\bigotimes_{j=1}^n V_{\sigma_j\beta}(x_j)\right], \tag{153}$$

and hence the expectation value (139) changes as

$$\begin{aligned}
&\omega^{\Lambda,\epsilon}\left(\delta\mathcal{T}\left[\bigotimes_{j=1}^n V_{\sigma_j\beta}(x_j)\otimes\mathcal{O}_{\mu\nu}(z)\right]\right)\\
&\quad= \eta_{\mu\nu}\sum_{j=1}^n c_{\sigma_j\beta}\delta^2(z-x_j)\,\omega^{\Lambda,\epsilon}\left(\mathcal{T}\left[\bigotimes_{j=1}^n V_{\sigma_j\beta}(x_j)\right]\right)\\
&\quad= \eta_{\mu\nu}\sum_{j=1}^n c_{\sigma_j\beta}\delta^2(z-x_j)\prod_{1\le i<j\le n}\left[\mu^2(-u_{ij}v_{ij}+\mathrm{i}\epsilon)\right]^{\sigma_i\sigma_j\frac{\beta^2}{4\pi}}\\
&\qquad\times\exp\left[-\frac{\beta^2}{2}\sum_{i,j=1}^n\sigma_i\sigma_j W(x_i,x_j)\right]\left(\frac{\Lambda}{\mu}\right)^{\frac{\beta^2\left(\sum_{j=1}^n\sigma_j\right)^2}{4\pi}}
\end{aligned} \tag{154}$$

using the result (131) and the explicit form of the Hadamard parametrix (118). Adding this correction to (139), we can cancel the divergent part by choosing

$$c_{\pm\beta} = \mathrm{i}\,\frac{\beta^2}{4\pi}\ln(2\mu\epsilon)\,. \tag{155}$$

We see clearly that the renormalisation is state-independent, which is one of the central insights of pAQFT. Moreover, as in the Euclidean case we only obtain a non-vanishing result in the limit where the IR cutoff $\Lambda \to 0$ if the sum of all σ_i vanishes, the super-selection criterion of the vacuum sector [45].

For the renormalised expectation value of the stress tensor $T_{\mu\nu} = \mathcal{O}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\mathcal{O}_\rho{}^\rho + g\eta_{\mu\nu}(V_\beta + V_{-\beta})$, we again obtain a sum of four terms, the first two of which can be read off from equations (139) and (154). As in the Euclidean case, since the divergent part in (150) is proportional to $\eta_{\mu\nu}$, it cancels out between the first two terms and the stress tensor is finite without counterterms. For the last two terms, we take the expectation value of equation (131) and use the explicit form of the Hadamard parametrix (118) to obtain

$$\begin{aligned}\omega^{\Lambda,\epsilon}\left(\mathcal{T}\left[\bigotimes_{j=1}^n V_{\sigma_j\beta}(x_j)\otimes V_\beta(z)\right]\right) &= \prod_{1\leq i<j\leq n}\left[\mu^2(-u_{ij}v_{ij}+\mathrm{i}\epsilon)\right]^{\sigma_i\sigma_j\frac{\beta^2}{4\pi}} \\ &\times\prod_{j=1}^n\left[\mu^2(-u(x_j,z)v(x_j,z)+\mathrm{i}\epsilon)\right]^{\sigma_j\frac{\beta^2}{4\pi}}\exp\left[-\frac{\beta^2}{2}\sum_{i,j=1}^n\sigma_i\sigma_j W(x_i,x_j)\right] \\ &\times\exp\left[-\beta^2\sum_{j=1}^n\sigma_j W(x_j,z)-\frac{\beta^2}{2}W(z,z)\right]\left(\frac{\Lambda}{\mu}\right)^{\frac{\beta^2\left(1+\sum_{j=1}^n\sigma_j\right)^2}{4\pi}}\ . \end{aligned} \tag{156}$$

Since we are in the finite regime $\beta^2 < 4\pi$, the singularities that arise for $\epsilon = 0$ as $x_j \to x_k$ and $x_j \to z$ are integrable, and so for this term no further renormalisation beyond the normal-ordering is required. Moreover, we again see how the neutrality condition appears: as $\Lambda \to 0$, we obtain a vanishing result unless $\sum_{j=1}^n \sigma_j = -1$. The last term with $V_{-\beta}$ results in the same result with σ_j replaced by $-\sigma_j$ on the right-hand side. □

3.3. Proof of theorem 5 (Convergence). As in the Euclidean case, we tacitly employ Fubini's theorem to interchange absolutely convergent integrals in this whole section, and consider numerator and denominator of equation (10) separately. Starting with the denominator, we use the result (131) from the proof of Lemma 2 to obtain

$$\begin{aligned}\omega^{0,0}\left(\mathcal{T}\left[\bigotimes_{j=1}^n V_{\sigma_j\beta}(x_j)\right]\right) &= \lim_{\Lambda,\epsilon\to 0}\left[\exp\left[-\mathrm{i}\beta^2\sum_{1\leq i<j\leq n}\sigma_i\sigma_j H^{\mathrm{F}}(x_i,x_j)\right]\right. \\ &\left.\times\exp\left[-\frac{\beta^2}{2}\sum_{i,j=1}^n\sigma_i\sigma_j W(x_i,x_j)\right]\left(\frac{\Lambda}{\mu}\right)^{\frac{\beta^2\left(\sum_{k=1}^n\sigma_k\right)^2}{4\pi}}\right] \\ &= \delta_{0,\sum_{k=1}^n\sigma_k}\prod_{1\leq i<j\leq n}\left[-\mu^2(u_{ij}v_{ij})_-\right]^{\sigma_i\sigma_j\frac{\beta^2}{4\pi}}\exp\left[-\frac{\beta^2}{2}\sum_{i,j=1}^n\sigma_i\sigma_j W(x_i,x_j)\right], \end{aligned} \tag{157}$$

where we use the notation $(uv)_\pm \equiv \lim_{\epsilon\to 0}(uv \pm \mathrm{i}\epsilon|u+v|)$ for the distributional boundary value. Since $\sigma_j = \pm 1$, to obtain a non-vanishing result we must have $n = 2m$ with m positive σ_j and m negative ones. We then rename the x_j with $\sigma_j = -1$ to y_j and renumber them. Taking into account that there are $\binom{n}{m} = (2m)!/(m!)^2$ possibilities to choose m positive σ_j from a total of $n = 2m$ ones (since equation (157) is symmetric under a permutation of the (renamed) x_i and y_j among themselves), the denominator of equation (10) reduces to

$$
\begin{aligned}
&\sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \int \cdots \int \omega^{0,0}\left(\mathcal{T}\left[\bigotimes_{j=1}^{m} V_\beta(x_j) \otimes V_{-\beta}(y_j)\right]\right) \prod_{i=1}^{m} g(x_i) g(y_i)\, \mathrm{d}^2 x_i\, \mathrm{d}^2 y_i \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \int \cdots \int \left[\frac{\prod_{1\le j<k\le m} [u(x_j,x_k) v(x_j,x_k)]_- [u(y_j,y_k) v(y_j,y_k)]_-}{(-1)^m \prod_{j,k=1}^{m} [u(x_j,y_k) v(x_j,y_k)]_-}\right]^{\frac{\beta^2}{4\pi}} \\
&\quad \times \mu^{-m\frac{\beta^2}{2\pi}} \exp\left[-\frac{\beta^2}{2} \sum_{i,j=1}^{m} [W(x_i,x_j) - W(y_i,x_j) - W(x_i,y_j) + W(y_i,y_j)]\right] \\
&\quad \times \prod_{i=1}^{m} g(x_i) g(y_i)\, \mathrm{d}^2 x_i\, \mathrm{d}^2 y_i\,.
\end{aligned} \tag{158}
$$

To bound the terms at order $2m$, we change the integration measure to light cone coordinates

$$
\mathrm{d}^2 x = \frac{1}{2}\, \mathrm{d}u(x)\, \mathrm{d}v(x)\,, \quad u(x) = x^0 - x^1\,, \quad v(x) = x^0 + x^1\,, \tag{159}
$$

and use that (115)

$$
u(x,y) = u(x) - u(y)\,, \quad v(x,y) = v(x) - v(y)\,. \tag{160}
$$

The absolute value of the terms in brackets factorises into a part depending on the u and a part depending on the v, and for each of them we use the Cauchy determinant formula:

$$
\begin{aligned}
\left|\frac{\prod_{1\le j<k\le m} u(x_j,x_k) u(y_j,y_k)}{\prod_{j,k=1}^{m} u(x_j,y_k)}\right|^p &= \left|\det\left(\frac{1}{u(x_i,y_j)}\right)_{i,j=1}^{m}\right|^p \\
&\le \left|\sum_\pi \prod_{j=1}^{m} \frac{1}{|u(x_j,y_{\pi(j)})|}\right|^p,
\end{aligned} \tag{161}
$$

where the sum runs over all permutations π of $\{1,\ldots,n\}$, and we used the estimate (59). For the exponential, we use the second assumption on W

$$
\sum_{i,j=1}^{m} [W(x_i,x_j) - W(y_i,x_j) - W(x_i,y_j) + W(y_i,y_j)] \ge 0\,, \tag{162}
$$

which lets us estimate the exponential by 1. It follows that the denominator (158) can be bounded by

$$\sum_{m=0}^{\infty} \frac{2^{-2m}}{(m!)^2} \int \cdots \int \left[\sum_{\pi} \prod_{j=1}^{m} \frac{1}{\mu \big| u(x_j, y_{\pi(j)}) \big|} \right]^{\frac{\beta^2}{4\pi}} \left[\sum_{\pi} \prod_{j=1}^{m} \frac{1}{\mu \big| v(x_j, y_{\pi(j)}) \big|} \right]^{\frac{\beta^2}{4\pi}} \\ \times \prod_{i=1}^{m} |g(x_i)||g(y_i)| \, \mathrm{d}u(x_i) \, \mathrm{d}v(x_i) \, \mathrm{d}u(y_i) \, \mathrm{d}v(y_i) \,. \tag{163}$$

We then estimate

$$|g(x_i)| \leq \left[1 + \mu^2 u(x_i)^2\right]^{-2} \left[1 + \mu^2 v(x_i)^2\right]^{-2} \\ \times \left\| \left[1 + \mu^2 u(x)^2\right]^2 \left[1 + \mu^2 v(x)^2\right]^2 g(x) \right\|_\infty , \tag{164}$$

such that the integrals over the u and the v factorise. We thus need to bound

$$\int \cdots \int \left[\sum_{\pi} \prod_{j=1}^{m} \frac{1}{\mu \big| u(x_j, y_{\pi(j)}) \big|} \right]^{\frac{\beta^2}{4\pi}} \prod_{i=1}^{m} \frac{\mathrm{d}u(x_i)}{\left[1 + \mu^2 u(x_i)^2\right]^2} \frac{\mathrm{d}u(y_i)}{\left[1 + \mu^2 u(y_i)^2\right]^2} , \tag{165}$$

and the analogous expression with u replaced by v. Using the Hölder inequality (73) with $r = \rho > 1$, this expression can be bounded by

$$\left[\int \cdots \int \left[\sum_{\pi} \prod_{j=1}^{m} \frac{1}{\mu \big| u(x_j, y_{\pi(j)}) \big|} \right]^{\rho \frac{\beta^2}{4\pi}} \prod_{i=1}^{m} \frac{\mathrm{d}u(x_i)}{[1 + \mu^2 u(x_i)^2]^\rho} \frac{\mathrm{d}u(y_i)}{[1 + \mu^2 u(y_i)^2]^\rho} \right]^{\frac{1}{\rho}} \\ \times \left[\int \cdots \int \prod_{i=1}^{m} \frac{\mathrm{d}u(x_i)}{\left[1 + \mu^2 u(x_i)^2\right]^{\frac{\rho}{\rho-1}}} \frac{\mathrm{d}u(y_i)}{\left[1 + \mu^2 u(y_i)^2\right]^{\frac{\rho}{\rho-1}}} \right]^{\frac{\rho-1}{\rho}} , \tag{166}$$

and choosing ρ such that $\rho \frac{\beta^2}{4\pi} < 1$ (which is possible since we are in the finite regime $\beta^2 < 4\pi$), we can use the estimate (60) to obtain

$$\left[\sum_{\pi} \prod_{j=1}^{m} \frac{1}{\mu \big| u(x_j, y_{\pi(j)}) \big|} \right]^{\rho \frac{\beta^2}{4\pi}} \leq \sum_{\pi} \prod_{j=1}^{m} \left[\mu \big| u(x_j, y_{\pi(j)}) \big| \right]^{-\rho \frac{\beta^2}{4\pi}} . \tag{167}$$

Since the remainder of the integrand is invariant under a permutation of the y_i, the sum over permutations π just gives a factor $m!$, so that we only need to bound

$$\iint \left[\mu |u(x_j, y_j)| \right]^{-\rho \frac{\beta^2}{4\pi}} \frac{\mathrm{d}u(x_j)}{[1 + \mu^2 u(x_j)^2]^\rho} \frac{\mathrm{d}u(y_j)}{[1 + \mu^2 u(y_j)^2]^\rho} . \tag{168}$$

In the region where $\mu |u(x_j, y_j)| > 1$, we estimate the first term by 1 and bound equation (168) by $\left[\int (1 + \mu^2 u^2)^{-\rho} \, \mathrm{d}u \right]^2 \leq \pi^2 \mu^{-2}$. In the region where $\mu |u(x_j, y_j)| \leq 1$, we use again Young's inequality in the form (64) with the exponents (65), but now in one dimension and with β^2 replaced by $\rho\beta^2$. This gives

$$\begin{aligned} &\left\|(1+\mu^2u^2)^{-\rho}\right\|_p^2 \left\| \Theta(1-\mu|u|)(\mu|u|)^{-\rho\frac{\beta^2}{4\pi}} \right\|_q \\ &= \mu^{-2}\left(\frac{\pi\Gamma\left(p\rho-\frac{1}{2}\right)}{\Gamma(p\rho)}\right)^{\frac{1}{p}}\left(\frac{8\pi}{4\pi-\rho\beta^2 q}\right)^{\frac{1}{q}}, \end{aligned} \tag{169}$$

where the result for the second norm is equation (67), and the first norm is a straightforward computation using [46, Eq. (5.12.3)] and [46, Eq. (5.12.1)]. The result is finite with the choice we made for p, q and ρ, in particular $\rho\beta^2 q = 4\pi - (4\pi-\rho\beta^2)(8\pi-\rho\beta^2)/(8\pi) < 4\pi$ and $p\rho > 1$. On the other hand, for the second factor in equation (166) we have the simple bound

$$\left[\int \frac{\mathrm{d}u}{\left(1+\mu^2u^2\right)^{\frac{\rho}{\rho-1}}}\right]^{\frac{\rho-1}{\rho}} \leq \rho\mu^{\frac{1-\rho}{\rho}}. \tag{170}$$

Taking all together, we can bound equation (166) (and thus equation (165)) by $(m!)^{\frac{1}{\rho}}\hat{K}^m$, where $\hat{K}$ is a constant depending on β and g, and we recall that $\rho > 1$. Inserting this result into equation (163), it follows that the denominator of equation (10) is bounded by

$$\sum_{m=0}^{\infty} \frac{2^{-2m}}{(m!)^2}(m!)^{\frac{2}{\rho}}\hat{K}^{2m} = \sum_{m=0}^{\infty}(m!)^{\frac{\beta^2}{4\pi}-1}K^m < \infty, \tag{171}$$

with the new constant $K = \hat{K}^2/4$, and where we made the (admissible) choice $\rho = 8\pi/(4\pi+\beta^2)$. We remark that the bounds (171) are not new and were already derived in [24]. However, a technical improvement over the proof of [24] is that we admit arbitrary adiabatic cutoff functions $g \in \mathcal{S}(\mathbb{R}^2)$, without any restriction on their support.

Consider thus the numerator of equation (10), where the renormalised expectation values are given by the sum of (139) and (154) with the choice (155) to cancel the divergent part. We see that in the physical limit $\Lambda \to 0$ again only even terms with $n = 2m$ contribute, and taking into account the symmetry under the exchange of variables and renaming integration variables as in the case of the denominator, the numerator reads

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \int f(z) \int \cdots \int \left[\frac{\prod_{1 \leq j < k \leq m} [u(x_j, x_k) v(x_j, x_k)]_- [u(y_j, y_k) v(y_j, y_k)]_-}{(-1)^m \prod_{j,k=1}^{m} [u(x_j, y_k) v(x_j, y_k)]_-} \right]^{\frac{\beta^2}{4\pi}} \\
&\quad \times \left[\lim_{z' \to z} \partial_\mu^z \partial_\nu^{z'} W(z, z') + \frac{\beta^2}{8\pi^2} \sum_{i,j=1}^{m} H_{(\mu}(z, x_i) H_{\nu)}(z, y_j) \right. \\
&\qquad - \beta^2 \sum_{i,j=1}^{m} \partial_\mu [W(z, x_i) - W(z, y_i)] \partial_\nu [W(z, x_j) - W(z, y_j)] \\
&\qquad + \frac{\beta^2}{2\pi} \sum_{i,j=1}^{m} [H_{(\mu}(z, x_i) - H_{(\mu}(z, y_i)] \partial_{\nu)} [W(z, x_j) - W(z, y_j)] \\
&\qquad - \frac{\beta^2}{8\pi^2} \sum_{1 \leq i < j \leq m} \big[H_{(\mu}(z, x_i) H_{\nu)}(z, x_j) + H_{(\mu}(z, y_i) H_{\nu)}(z, y_j) \big] \\
&\qquad \left. - \frac{\beta^2}{16\pi^2} \sum_{j=1}^{m} \Big[[H_\mu(z, x_j) H_\nu(z, x_j)]^{\text{ren}} + [H_\mu(z, y_j) H_\nu(z, y_j)]^{\text{ren}} \Big] \right] \\
&\quad \times \mu^{-m \frac{\beta^2}{2\pi}} \exp \left[-\frac{\beta^2}{2} \sum_{i,j=1}^{m} [W(x_i, x_j) - W(y_i, x_j) - W(x_i, y_j) + W(y_i, y_j)] \right] \\
&\quad \times \mathrm{d}^2 z \prod_{i=1}^{m} g(x_i) g(y_i) \, \mathrm{d}^2 x_i \, \mathrm{d}^2 y_i \, . \qquad (172)
\end{aligned}$$

We see that there are various different types of terms, some of which are equal since we can interchange x_j and y_j without changing the result, and we will bound all of them separately. Consider first the terms that only contain W but no derivatives H_μ of the Hadamard parametrix. By the first assumption on W, it and its derivatives grow at most polynomially such that

$$\left| \lim_{z' \to z} \partial_\mu^z \partial_\nu^{z'} W(z, z') \right| \leq w \Big[1 + \mu^2 u(z)^2 + \mu^2 v(z)^2 \Big]^k , \tag{173a}$$

$$|\partial_\mu W(z, x)| \leq w \Big[1 + \mu^2 u(z)^2 + \mu^2 v(z)^2 \Big]^k \Big[1 + \mu^2 u(x)^2 + \mu^2 v(x)^2 \Big]^k \tag{173b}$$

for some constant $w > 0$ and some $k \in \mathbb{N}$. It follows that the contribution of the terms that only contain W but no derivatives H_μ to the numerator (172) can be bounded by

$$\sum_{m=0}^{\infty} \frac{\mu^{-m\frac{\beta^2}{2\pi}}}{(m!)^2} \int |f(z)| \int \cdots \int \left| \frac{\prod_{1\leq j<k\leq m} u(x_j,x_k)v(x_j,x_k)u(y_j,y_k)v(y_j,y_k)}{\prod_{j,k=1}^{m} u(x_j,y_k)v(x_j,y_k)} \right|^{\frac{\beta^2}{4\pi}}$$
$$\times \left[w\Big[1+\mu^2 u(z)^2+\mu^2 v(z)^2\Big]^k + \beta^2 w\Big[1+\mu^2 u(z)^2+\mu^2 v(z)^2\Big]^{2k} \right.$$
$$\left. \times \left[\sum_{i=1}^{m} \Big[1+\mu^2 u(x_i)^2+\mu^2 v(x_i)^2\Big]^k + \Big[1+\mu^2 u(y_i)^2+\mu^2 v(y_i)^2\Big]^k \right]^2 \right]$$
$$\times \exp\left[-\frac{\beta^2}{2} \sum_{i,j=1}^{m} [W(x_i,x_j)-W(y_i,x_j)-W(x_i,y_j)+W(y_i,y_j)] \right]$$
$$\times \mathrm{d}^2 z \prod_{i=1}^{m} |g(x_i)||g(y_i)| \, \mathrm{d}^2 x_i \, \mathrm{d}^2 y_i \,. \tag{174}$$

The integral over z can be estimated by a constant C depending on the test function f and the constants w, k and β^2. Absorbing the terms $\Big[1+\mu^2 u(x_i)^2 + \mu^2 v(x_i)^2\Big]^k$ into the test functions $g(x_i)$ to obtain new test functions $\tilde{g}$, and taking into account that the middle sum in equation (174) contributes m^2 terms, which by renaming of integration variables all give the same contribution, we can then repeat the derivation of the denominator estimates for the remaining terms. It follows that equation (174) is bounded by

$$C \sum_{m=0}^{\infty} (m!)^{\frac{\beta^2}{4\pi}-1} m^2 K^m < \infty \,, \tag{175}$$

where K depends on $\tilde{g}$ and thus also on W. Next consider the mixed terms involving W and $H_\mu = -4\pi\mathrm{i}\partial_\mu H^{\mathrm{F}}$ (140). Integrating the derivative by parts and using the estimates (173) for W, the contribution of these terms to the numerator (172) can be bounded by

$$\sum_{m=0}^{\infty} \frac{\mu^{-m\frac{\beta^2}{2\pi}}}{(m!)^2} \int \int \cdots \int \left| \frac{\prod_{1\leq j<k\leq m} u(x_j,x_k)v(x_j,x_k)u(y_j,y_k)v(y_j,y_k)}{\prod_{j,k=1}^{m} u(x_j,y_k)v(x_j,y_k)} \right|^{\frac{\beta^2}{4\pi}}$$
$$\times 8m^2 w\beta^2 \left[|f(z)| + \sup_\mu |\partial_\mu f(z)| \right] \Big[1+\mu^2 u(z)^2+\mu^2 v(z)^2\Big]^k \left|H^{\mathrm{F}}(z,x_1)\right|$$
$$\times \exp\left[-\frac{\beta^2}{2} \sum_{i,j=1}^{m} [W(x_i,x_j)-W(y_i,x_j)-W(x_i,y_j)+W(y_i,y_j)] \right]$$
$$\times \mathrm{d}^2 z \prod_{i=1}^{m} |\tilde{g}(x_i)||\tilde{g}(y_i)| \, \mathrm{d}^2 x_i \, \mathrm{d}^2 y_i \,, \tag{176}$$

where we have again absorbed terms $\left[1 + \mu^2 u(x_i)^2 + \mu^2 v(x_i)^2\right]^k$ into the test functions $g(x_i)$, and the factor of m^2 arises because all the terms in the sum give the same contribution. To compute the integral over z, we use that the Hadamard parametrix $\left|H^\mathrm{F}\right|$ (118) factors in the limit $\epsilon \to 0$, such that

$$\left|H^\mathrm{F}(z,x)\right| \leq \frac{1}{4\pi} \left|\ln\left|\mu u(z,x)\right|\right| + \frac{1}{4\pi} \left|\ln\left|\mu v(z,x)\right|\right| + \frac{1}{4} \,. \tag{177}$$

The contribution of the last term is bounded by

$$\int 2w\beta^2 \left[\left|f(z)\right| + \sup_\mu \left|\partial_\mu f(z)\right| \right] \left[1 + \mu^2 u(z)^2 + \mu^2 v(z)^2\right]^k \mathrm{d}^2 z \leq C \,, \tag{178}$$

where the constant C depends on f and W through w and k. For the contributions of the logarithms in (177), we first bound

$$\begin{split} &8w\beta^2 \left[\left|f(z)\right| + \sup_\mu \left|\partial_\mu f(z)\right| \right] \left[1 + \mu^2 u(z)^2 + \mu^2 v(z)^2\right]^k \\ &\quad \leq C \left[1 + \mu^2 u(z)^2\right]^{-1} \left[1 + \mu^2 u(z)^2\right]^{-1} \,, \end{split} \tag{179}$$

where the constant C depends on f and W through w and k. We then change the z integration to light cone coordinates (159), such that the integral over z factors, and then use the estimate

$$\int \frac{\left|\ln\left|\mu u(z,x)\right|\right|^k}{1 + \mu^2 u(z)^2} \,\mathrm{d}u(z) \leq \frac{2}{\mu} c_k \ln^k\left(2 + \mu\left|u(x)\right|\right) . \tag{180}$$

and the analogous one with u replaced by v. It follows that the integral over z in equation (176) is bounded by a constant C depending on f, g and W. For the remaining terms we repeat the derivation of the denominator estimates, absorbing the logarithm of equation (180) in the test functions g since $\left|g(x_i)\right| \ln(2 + \mu\left|u(x_i)\right|)$ is still a rapidly decreasing function. It follows that also the contribution of the terms involving both W and H_μ to the numerator (172) is bounded by a sum of the form (175).

The bound (180) is proven as follows: we estimate first that

$$\begin{split} \int \frac{\left|\ln\left|\mu u(z,x)\right|\right|^k}{1 + \mu^2 u(z)^2} \,\mathrm{d}u(z) &= \frac{1}{\mu} \int \frac{\left|\ln\left|s\right|\right|^k}{1 + [s + \mu u(x)]^2} \,\mathrm{d}s \\ &\leq \frac{2}{\mu} \int_0^\infty \frac{\left|\ln s\right|^k}{1 + [s - \mu\left|u(x)\right|]^2} \,\mathrm{d}s \,, \end{split} \tag{181}$$

and thus only have to show that

$$\int_0^\infty \frac{\left|\ln s\right|^k}{1 + (s-a)^2} \,\mathrm{d}s \leq c_k \ln^k(2 + a) \tag{182}$$

for $a \geq 0$. We compute

$$\begin{aligned}\int_0^\infty \frac{|\ln s|^k}{1+(s-a)^2}\,\mathrm{d}s &= \int_0^1 \frac{(-\ln s)^k}{1+(s-a)^2}\,\mathrm{d}s + \int_1^\infty \frac{\ln^k s}{1+(s-a)^2}\,\mathrm{d}s \\ &\leq \int_0^1 (-\ln s)^k\,\mathrm{d}s + \int_1^{1+a} \frac{\ln^k s}{1+(s-a)^2}\,\mathrm{d}s + \int_{1+a}^\infty \frac{\ln^k s}{1+(s-a)^2}\,\mathrm{d}s \\ &\leq k! + \ln^k(1+a) \int_1^{1+a} \frac{1}{1+(s-a)^2}\,\mathrm{d}s + \int_1^\infty \frac{\ln^k(s+a)}{1+s^2}\,\mathrm{d}s \\ &\leq k! + \frac{3}{4}\pi \ln^k(1+a) + \int_1^\infty \frac{[\ln(s)+\ln(2+a)]^k}{1+s^2}\,\mathrm{d}s\end{aligned} \tag{183}$$

using the inequality (valid for $a \geq 0$, $b \geq 1$)

$$\ln(a+b) \leq \ln b + \ln(2+a) \quad \Leftrightarrow \quad (a+b) \leq b(2+a)\,, \tag{184}$$

which is proven in the standard way by showing that it holds for $a = 0$ and that the first derivative with respect to a of the left-hand side is smaller than the one on the right-hand side. Using further that

$$\int_1^\infty \frac{\ln^k(s)}{1+s^2}\,\mathrm{d}s \leq k!\,, \tag{185}$$

we obtain

$$\begin{aligned}\int_0^\infty \frac{|\ln s|^k}{1+(s-a)^2}\,\mathrm{d}s &\leq k! + \frac{3}{4}\pi \ln^k(1+a) + \sum_{m=0}^k \binom{k}{m} \ln^{k-m}(2+a) m! \\ &\leq \left(2k! + \frac{3}{4}\pi\right) \ln^k(2+a)\end{aligned} \tag{186}$$

as required.

The remaining terms in equation (172) either involve two derivatives H_μ of the Hadamard parametrix at different points, or the renormalised product $[H_\mu(z,x)H_\nu(z,x)]^{\text{ren}}$ (151). We start with the latter ones, which contain local terms proportional to $\delta^2(z-x)$ as well as derivatives acting on the Hadamard parametrix and its square. The local terms allow to perform the integral over z, resulting in a factor $f(x_i)$ or $f(y_j)$ which can be estimated by $\|f\|_\infty$, and for the remaining terms we can repeat the derivation of the denominator estimates. The terms with derivatives acting on the Hadamard parametrix and its square are integrated by parts to act on f, and as before we introduce factors of $[1+\mu^2 u(z)^2]$ and $[1+\mu^2 v(z)^2]$ and use that $\left|[1+\mu^2 u(z)^2][1+\mu^2 v(z)^2]\partial_\mu \partial_\nu f(z)\right| \leq C$. The remaining integral over z is then bounded using equation (180). We can again absorb the logarithm in the test functions g, and since there are $2m \leq 2m^2$ terms involving $[H_\mu(z,x)H_\nu(z,x)]^{\text{ren}}$, it follows that also their contribution to the numerator (172) is bounded by a sum of the form (175).

To bound the remaining terms with two derivatives H_μ of the Hadamard parametrix at different points, we first consider the case where $\mu = u$, $\nu = v$.

Using that H^F is a fundamental solution of the Klein–Gordon equation (119) and the definition of H_μ (140), a straightforward computation results in

$$\begin{split}\int H_{(u}(z,x) H_{v)}(z,y) f(z) \,\mathrm{d}^2 z = -2\pi^2 H^\mathrm{F}(x,y) [f(x) + f(y)] \\ - 8\pi^2 \int H^\mathrm{F}(z,x) H^\mathrm{F}(z,y) \partial_u \partial_v f(z) \,\mathrm{d}^2 z \,. \end{split} \tag{187}$$

To estimate the second term, we introduce factors of $[1+\mu^2 u(z)^2]$ and $[1+\mu^2 v(z)^2]$ and use Hölder's inequality (73) with $r = 2$ to obtain

$$\begin{split}\left\lvert \int H^\mathrm{F}(z,x) H^\mathrm{F}(z,y) \partial_u \partial_v f(z) \,\mathrm{d}^2 z \right\rvert \leq \left\lVert [1+\mu^2 u(z)^2] [1+\mu^2 v(z)^2] \partial_u \partial_v f \right\rVert_\infty \\ \times \left[\int \frac{\left\lvert H^\mathrm{F}(z,x) \right\rvert^2}{[1+\mu^2 u(z)^2] [1+\mu^2 v(z)^2]} \,\mathrm{d}^2 z \int \frac{\left\lvert H^\mathrm{F}(z,y) \right\rvert^2}{[1+\mu^2 u(z)^2] [1+\mu^2 v(z)^2]} \,\mathrm{d}^2 z \right]^\frac{1}{2} \,. \end{split} \tag{188}$$

Using the bound (177) for the Hadamard parametrix and the estimate (180), we obtain the bound

$$\begin{split}\left\lvert \int H^\mathrm{F}(z,x) H^\mathrm{F}(z,y) \partial_u \partial_v f(z) \,\mathrm{d}^2 z \right\rvert \leq C \ln(2+\mu \lvert u(x) \rvert) \ln(2+\mu \lvert v(x) \rvert) \\ \times \ln(2+\mu \lvert u(y) \rvert) \ln(2+\mu \lvert v(y) \rvert) \,, \end{split} \tag{189}$$

where the constant C depends on f, and we can again absorb the logarithms in the test functions g. Repeating the derivation of the denominator estimates and taking into account that there are $m^2 + m(m-1) < 2m^2$ terms with two derivatives H_μ of the Hadamard parametrix at different points, it follows that also the contribution of the second term in equation (187) to the numerator (172) is bounded by a sum of the form (175). On the other hand, the first term in equation (187) is directly bounded using the bound (177) for the Hadamard parametrix, which is however logarithmically divergent for small $u(x,y)$ or $v(x,y)$. We can then almost repeat the derivation of the denominator estimates, except that we need to bound equation (168) in the case that an additional logarithm is present, i.e., we need to bound

$$\iint \left[\mu \lvert u(x_j,y_j) \rvert \right]^{-\rho \frac{\beta^2}{4\pi}} \lvert \ln \lvert \mu u(x_j,y_k) \rvert \rvert \frac{\mathrm{d} u(x_j)}{[1+\mu^2 u(x_j)^2]^\rho} \frac{\mathrm{d} u(y_j)}{[1+\mu^2 u(y_j)^2]^\rho} \tag{190}$$

in the two cases $k = j$ and $k \neq j$. We start with the case $k = j$, and use the well-known bound[1]

$$\ln z \leq \frac{1}{r} (z^r - 1) \leq \frac{1}{r} z^r \tag{191}$$

for $z, r \geq 0$ to estimate

$$\left[\mu \lvert u(x_j,y_j) \rvert \right]^{-\rho \frac{\beta^2}{4\pi}} \lvert \ln \lvert \mu u(x_j,y_j) \rvert \rvert \leq \frac{8\pi}{\rho \beta^2} \left[\mu \lvert u(x_j,y_j) \rvert \right]^{-\rho \frac{\beta^2}{8\pi}} \,. \tag{192}$$

[1] It can be proven in the standard way, noting that for $f(z) = \ln z - \frac{1}{r}(z^r - 1)$ we have $f(1) = 0$ and $z f'(z) = 1 - z^r \leq 0$ for $z \geq 1$, $z f'(z) \geq 0$ for $z \leq 1$.

In the region $\mu|u(x_j,y_j)| > 1$, we then estimate this term by $8\pi/(\rho\beta^2)$, and bound equation (190) by $8\pi/(\rho\beta^2)\big[\int(1+\mu^2u^2)^{-\rho}\,\mathrm{d}u\big]^2 \leq 8\pi^3/(\rho\beta^2\mu^2)$. In the region where $\mu|u(x_j,y_j)| \leq 1$, we use again Young's inequality in the form (64) with the exponents (65), but now in one dimension and with β^2 replaced by $\rho\beta^2/2$. This gives

$$
\begin{aligned}
&\big\|(1+\mu^2u^2)^{-\rho}\big\|_p^2 \left\| \Theta(1-\mu|u|)(\mu|u|)^{-\rho\frac{\beta^2}{8\pi}} \right\|_q \\
&\quad = \mu^{-2} \left(\frac{\pi\Gamma\left(\frac{p\rho-1}{2}\right)}{\Gamma\left(\frac{p\rho}{2}\right)} \right)^{\frac{1}{p}} \left(\frac{16\pi}{8\pi-\rho\beta^2 q} \right)^{\frac{1}{q}} ,
\end{aligned}
\tag{193}
$$

which is finite with the choice we made for p, q and ρ as before. For $k \neq j$, we use again Hölder's inequality (73) with $r = (4\pi-\rho\beta^2)/(2\rho\beta^2)$ to obtain

$$
\begin{aligned}
&\iint \Big[\mu|u(x_j,y_j)|\Big]^{-\rho\frac{\beta^2}{4\pi}} |\ln|\mu u(x_j,y_k)|| \frac{\mathrm{d}u(x_j)}{[1+\mu^2u(x_j)^2]^\rho} \frac{\mathrm{d}u(y_j)}{[1+\mu^2u(y_j)^2]^\rho} \\
&\quad \leq \left[\iint \Big[\mu|u(x_j,y_j)|\Big]^{-r\rho\frac{\beta^2}{4\pi}} \frac{\mathrm{d}u(x_j)}{[1+\mu^2u(x_j)^2]^\rho} \frac{\mathrm{d}u(y_j)}{[1+\mu^2u(y_j)^2]^\rho} \right]^{\frac{1}{r}} \\
&\quad\quad \times \left[\iint |\ln|\mu u(x_j,y_k)||^{\frac{r}{r-1}} \frac{\mathrm{d}u(x_j)}{[1+\mu^2u(x_j)^2]^\rho} \frac{\mathrm{d}u(y_j)}{[1+\mu^2u(y_j)^2]^\rho} \right]^{\frac{r-1}{r}} .
\end{aligned}
\tag{194}
$$

With this choice of r, we have $r > 1$ and $r\rho\beta^2/(4\pi) < 1$ such that both integrals are convergent, and we can bound each of them by repeating the estimates used to bound (190). Since there are $m^2 + m(m-1) < 2m^2$ terms with two derivatives H_μ of the Hadamard parametrix at different points, it follows that also the contribution of the first term in equation (187) to the numerator (172) is bounded by a sum of the form (175).

We thus consider the remaining terms with two derivatives of the Hadamard parametrix at different points for $\mu = \nu = u$; the case $\mu = \nu = v$ is completely analogous. Using the definition of H_μ (140), we compute

$$
\begin{aligned}
H_u(z,x)H_u(z,y) &= \sum_{a,b=\pm} \Theta(a(u+v)(z,x))\Theta(b(u+v)(z,y)) \\
&\qquad \times \frac{\partial}{\partial u(z)} \frac{\ln[(u(z,x)-a\mathrm{i}\epsilon)^2] - \ln[(u(z,y)-b\mathrm{i}\epsilon)^2]}{2u(x,y)+2(a-b)\mathrm{i}\epsilon} \\
&= \left[\frac{\partial}{\partial u(z)} - \frac{\partial}{\partial v(z)} \right] \sum_{a,b=\pm} \Theta(a(u+v)(z,x))\Theta(b(u+v)(z,y)) \\
&\qquad \times \frac{\ln[(u(z,x)-a\mathrm{i}\epsilon)^2] - \ln[(u(z,y)-b\mathrm{i}\epsilon)^2]}{2u(x,y)+2(a-b)\mathrm{i}\epsilon} \\
&= \left[\frac{\partial}{\partial u(z)} - \frac{\partial}{\partial v(z)} \right]^2 \sum_{a,b=\pm} \Theta(a(u+v)(z,x))\Theta(b(u+v)(z,y)) \\
&\qquad \times \frac{(u(z,x)-a\mathrm{i}\epsilon)\ln[(u(z,x)-a\mathrm{i}\epsilon)^2] - (u(z,y)-b\mathrm{i}\epsilon)\ln[(u(z,y)-b\mathrm{i}\epsilon)^2]}{2u(x,y)+2(a-b)\mathrm{i}\epsilon} ,
\end{aligned}
\tag{195}
$$

and since the last fraction is integrable in $u(z)$ and has a finite limit as $y \to x$ even for $\epsilon = 0$, we can take the limit $\epsilon \to 0$ and obtain

$$H_u(z,x) H_u(z,y) = \frac{\partial^2}{\partial u(z)^2} \left[\frac{u(z,x) \ln[u(z,x)^2] - u(z,y) \ln[u(z,y)^2]}{2u(x,y)} \right] . \quad (196)$$

Using the mean value theorem for $f(u(x)) = u(z,x) \ln[u(z,x)^2]$, we have

$$\frac{u(z,x) \ln[u(z,x)^2] - u(z,y) \ln[u(z,y)^2]}{u(x,y)} = f'(u(a)) = -2 - \ln[u(z,a)^2] \quad (197)$$

for some point a such that $\min(u(x), u(y)) \leq u(a) \leq \max(u(x), u(y))$, and thus

$$H_u(z,x) H_u(z,y) = -\frac{1}{2} \frac{\partial^2}{\partial u(z)^2} \ln\left[\mu^2 u(z,a)^2 \right] . \quad (198)$$

Introducing as before factors $[1 + \mu^2 u(z)^2]$, it follows that we can estimate

$$\begin{aligned} \left| \int H_u(z,x) H_u(z,y) f(z) \, \mathrm{d}^2 z \right| &\leq \frac{1}{2} \left\| [1 + \mu^2 u(z)^2][1 + \mu^2 v(z)^2] \partial_u^2 f(z) \right\|_\infty \\ &\quad \times \int \lvert \ln \lvert u(z,a) \rvert \rvert \frac{\mathrm{d}u(z)}{1 + \mu^2 u(z)^2} \int \frac{\mathrm{d}v(z)}{1 + \mu^2 v(z)^2} , \end{aligned} \quad (199)$$

and the integrals are estimated using equation (180). Finally, we use

$$\ln(2 + \mu \lvert u(a) \rvert) \leq \ln(2 + \mu \lvert u(x) \rvert) + \ln(2 + \mu \lvert u(y) \rvert) , \quad (200)$$

absorb the logarithms in the test functions g and repeat the derivation of the denominator estimates. It follows that also the contribution of the terms with two derivatives of the Hadamard parametrix at different points for $\mu = \nu = u$ and $\mu = \nu = v$ to the numerator (172) is bounded by a sum of the form (175), using again that we have $m^2 + m(m-1) < 2m^2$ of this type.

For the first two terms of the stress tensor $T_{\mu\nu} = \mathcal{O}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \mathcal{O}_\rho{}^\rho + g \eta_{\mu\nu} (V_\beta + V_{-\beta})$ we can take over the above bounds. For the third term, we use the result (156) and thus have to bound

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{n!} \int \cdots \int \sum_{\sigma_i=\pm 1} \omega^{0,0}\left(\mathcal{T}\left[V_\beta(gf) \otimes \bigotimes_{j=1}^{n} V_{\sigma_j \beta}(x_j) \right] \right) \prod_{i=1}^{n} g(x_i)\, \mathrm{d}^2 x_i \\
&= \mathrm{i} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)!} \mu^{-(m+1)\frac{\beta^2}{2\pi}} \int f(z) g(z) \int \cdots \int \\
&\quad \times \left[\frac{\prod_{1\le j<k\le m} [u(x_j,x_k)v(x_j,x_k)]_- \prod_{1\le j<k\le m+1} [u(y_j,y_k)v(y_j,y_k)]_-}{(-1)^m \prod_{j=1}^{m} \prod_{k=1}^{m+1} [u(x_j,y_k)v(x_j,y_k)]_-} \right]^{\frac{\beta^2}{4\pi}} \\
&\quad \times \left[\frac{1}{-[u(y_{m+1},z)v(y_{m+1},z)]_-} \prod_{j=1}^{m} \frac{[u(x_j,z)v(x_j,z)]_-}{[u(y_j,z)v(y_j,z)]_-} \right]^{\frac{\beta^2}{4\pi}} \\
&\quad \times \exp\left[-\frac{\beta^2}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} [W(x_i,x_j) - W(y_i,x_j) - W(x_i,y_j) + W(y_i,y_j)] \right] \\
&\quad \times \exp\left[-\beta^2 \sum_{i=1}^{m} [W(x_i,z) - W(y_i,z) - W(x_i,y_{m+1}) + W(y_i,y_{m+1})] \right] \\
&\quad \times \exp\left[-\frac{\beta^2}{2} [W(y_{m+1},y_{m+1}) - 2W(y_{m+1},z) + W(z,z)] \right] \\
&\quad \times \mathrm{d}^2 z\, \mathrm{d}^2 y_{m+1} \prod_{i=1}^{m} g(x_i) g(y_i)\, \mathrm{d}^2 x_i\, \mathrm{d}^2 y_i\,,
\end{aligned} \tag{201}
$$

where we used that because of the neutrality condition only odd terms $n = 2m+1$ give a non-vanishing contribution. Of these, m have a positive σ_j and $m+1$ have a negative one, such that the sum over the σ_j resulted in a factor of $\binom{2m+1}{m} = (2m+1)!/(m!(m+1)!)$, and as before we renamed the integration variables with a negative σ_j to y_j. The second assumption on W shows that the exponentials can be bounded by 1, and setting $x_{m+1} \equiv z$ the terms in brackets combine to

$$
\left[\frac{\prod_{1\le j<k\le m+1} [u(x_j,x_k)v(x_j,x_k)]_- [u(y_j,y_k)v(y_j,y_k)]_-}{(-1)^{m+1} \prod_{j,k=1}^{m+1} [u(x_j,y_k)v(x_j,y_k)]_-} \right]^{\frac{\beta^2}{4\pi}} . \tag{202}
$$

We can then use the same steps as in bounding the denominator (158), with the result that the series (201) is bounded by

$$
\|f\|_\infty \sum_{m=0}^{\infty} \frac{1}{m!(m+1)!} [(m+1)!]^{1+\frac{\beta^2}{4\pi}} K^{m+1} \le C \sum_{m=0}^{\infty} (m!)^{\frac{\beta^2}{4\pi}-1} m^2 K^m\,, \tag{203}
$$

with the constant C now also depending on K and thus on g. The same bound is obtained analogously for the fourth term in the stress tensor involving $V_{-\beta}$, which switches x_i with y_i in equation (201).

Since we have shown that the denominator of the Gell-Mann–Low formula (10) is a convergent series in g (171) starting with 1, and is thus bounded by $\frac{1}{2}$ from below for sufficiently small $\|g\|_\infty$, the required bounds for the expectation values of $\mathcal{O}_{\mu\nu}(f)$ and $T_{\mu\nu}(f)$ follow. □

3.4. Proof of theorem 6 (Conservation). As in the Euclidean case, to show that the interacting stress tensor is conserved it is enough to show that the numerator of the Gell-Mann–Low formula vanishes when smeared with a test function of the form $\partial^\mu f$ with $f \in \mathcal{S}(\mathbb{R}^2)$. Consider the numerator for $\mathcal{O}_{\mu\nu}$ (172), and smear it with $\partial^\mu f$. The result contains now three different types of terms: the ones with the renormalised $[H_\mu(x,y)H_\nu(x,y)]^{\text{ren}}$ (151), the ones involving double sums of H_μ, and the ones involving W. We start with the latter type, which contains double sums involving W, a coincidence limit of derivatives of W, and a mixed double sum involving W and H_μ. For the double sums involving W, we compute

$$\begin{aligned} &\int \partial^\mu f(z) \left[\partial_{(\mu} W(z,x) \partial_{\nu)} W(z,y) - \frac{1}{2} \eta_{\mu\nu} \partial^\rho W(z,x) \partial_\rho W(z,y) \right] \mathrm{d}^2 z \\ &\qquad = -\frac{1}{2} \int f(z) \left[\partial^2 W(z,x) \partial_\nu W(z,y) + \partial_\nu W(z,x) \partial^2 W(z,y) \right] \mathrm{d}^2 z \,, \end{aligned} \tag{204}$$

where x and y do not need to be distinct. Since W is a bisolution of the Klein–Gordon equation $\partial_x^2 W(x,y) = \partial_y^2 W(x,y) = 0$, these terms vanish. For the terms involving the coincidence limit, we use Synge's rule [61]

$$\partial^z_\mu \lim_{z' \to z} f(z,z') = \lim_{z' \to z} \left[\partial^z_\mu f(z,z') + \partial^{z'}_\mu f(z,z') \right] , \tag{205}$$

and compute

$$\begin{aligned} &\int \partial^\mu f(z) \lim_{z' \to z} \left[\partial^z_\mu \partial^{z'}_\nu W(z,z') - \frac{1}{2} \eta_{\mu\nu} \partial^z_\rho \partial^\rho_{z'} W(z,z') \right] \mathrm{d}^2 z \\ &\qquad = - \int f(z) \lim_{z' \to z} \left[\partial^2_z \partial^{z'}_\nu W(z,z') - \frac{1}{2} \partial^z_\rho \left(\partial^z_\nu - \partial^{z'}_\nu \right) \partial^\rho_{z'} W(z,z') \right] \mathrm{d}^2 z \,. \end{aligned} \tag{206}$$

The first term again vanishes since W is a bisolution, while the second one vanishes because $W(x,y) = W(y,x)$ is symmetric in x and y, exchanging z and z' in one of the two parts. Lastly, for the mixed terms involving both W and H_μ, we obtain

$$\begin{aligned} &\int \partial^\mu f(z) \left[H_{(\mu}(z,x) \partial_{\nu)} W(z,y) - \frac{1}{2} \eta_{\mu\nu} H^\rho(z,x) \partial_\rho W(z,y) \right] \mathrm{d}^2 z \\ &\qquad = - \int f(z) \left[\frac{1}{2} \partial^\mu H_\mu(z,x) \partial_\nu W(z,y) + \frac{1}{2} H_\nu(z,x) \partial^2 W(z,y) \right] \mathrm{d}^2 z \,, \end{aligned} \tag{207}$$

and since W is a bisolution, the last term vanishes. However, for $\partial^\mu H_\mu$ we obtain instead using the definition of H_μ (140)

$$\partial^\mu H_\mu(z,x) = -4\pi\mathrm{i}\, \partial^2 H^{\mathrm{F}}(z,x) = -4\pi\mathrm{i}\, \delta^2(z-x) \,, \tag{208}$$

since the Hadamard parametrix H^{F} is not a bisolution of the (massless) Klein–Gordon equation, but instead a fundamental solution (119), and hence

$$\begin{aligned}&\int \partial^\mu f(z) \left[H_{(\mu}(z,x)\partial_{\nu)} W(z,y) - \frac{1}{2}\eta_{\mu\nu} H^\rho(z,x)\partial_\rho W(z,y) \right] \mathrm{d}^2 z \\ &\quad = 2\pi\mathrm{i} f(x)\, \partial_\nu W(x,y)\,.\end{aligned} \tag{209}$$

Similarly, for the terms involving double sums containing $H_{(\mu}(z,x_i)H_{\nu)}(z,x_j)$ with $i \neq j$, we obtain

$$\begin{aligned}&\int \partial^\mu f(z) \left[H_{(\mu}(z,x)H_{\nu)}(z,y) - \frac{1}{2}\eta_{\mu\nu} H^\rho(z,x)H_\rho(z,y) \right] \mathrm{d}^2 z \\ &\quad = 2\pi\mathrm{i}\,[f(x)\, H_\nu(x,y) + f(y)\, H_\nu(y,x)] = 2\pi\mathrm{i}\,[f(x) - f(y)] H_\nu(x,y)\,,\end{aligned} \tag{210}$$

and for the terms involving the renormalised $[H_\mu(x,y)H_\nu(x,y)]^{\mathrm{ren}}$ it follows that

$$\begin{aligned}&\int \partial^\mu f(z) \left[[H_\mu(z,x)H_\nu(z,x)]^{\mathrm{ren}} - \frac{1}{2}\eta_{\mu\nu} [H^\rho(z,x)H_\rho(z,x)]^{\mathrm{ren}} \right] \mathrm{d}^2 z \\ &\quad = -2\delta^u_\nu \int \partial_v f(z) [H_u(z,x)H_u(z,x)]^{\mathrm{ren}}\, \mathrm{d}^2 z \\ &\qquad - 2\delta^v_\nu \int \partial_u f(z) [H_v(z,x)H_v(z,x)]^{\mathrm{ren}}\, \mathrm{d}^2 z\end{aligned} \tag{211}$$

using the form of the Minkowski metric in light cone coordinates (149). Inserting the explicit expressions for the renormalised $[H_\mu(x,y)H_\nu(x,y)]^{\mathrm{ren}}$ (151), we obtain

$$\begin{aligned}&\int \partial^\mu f(z) \left[[H_\mu(z,x)H_\nu(z,x)]^{\mathrm{ren}} - \frac{1}{2}\eta_{\mu\nu} [H^\rho(z,x)H_\rho(z,x)]^{\mathrm{ren}} \right] \mathrm{d}^2 z \\ &\quad = -8\pi\mathrm{i} \int \partial_u \partial_v \partial_\nu f(z) H^{\mathrm{F}}(z,x)\, \mathrm{d}^2 z - 2\pi\mathrm{i}[\delta^u_\nu \partial_v f(x) + \delta^v_\nu \partial_u f(x)] \\ &\quad = 2\pi\mathrm{i}\, \partial_\nu f(x) - 2\pi\mathrm{i}[\delta^u_\nu \partial_v f(x) + \delta^v_\nu \partial_u f(x)]\,,\end{aligned} \tag{212}$$

using that H^{F} is a fundamental solution (119) of the Klein–Gordon equation.

Analogous to the Euclidean case (96), we compute

$$\begin{aligned}&\partial^{x_\ell}_\nu \ln\left(\left[\frac{\prod_{1\leq j<k\leq m} [u(x_j,x_k)v(x_j,x_k)]_- [u(y_j,y_k)v(y_j,y_k)]_-}{(-1)^m \prod_{j,k=1}^m [u(x_j,y_k)v(x_j,y_k)]_-} \right]^{\frac{\beta^2}{4\pi}} \right) \\ &\quad = \delta^u_\nu \frac{\beta^2}{4\pi} \left[\sum_{k\neq\ell} \frac{1}{[u(x_\ell,x_k)]_{-\operatorname{sgn}(u+v)}} - \sum_{k=1}^m \frac{1}{[u(x_\ell,y_k)]_{-\operatorname{sgn}(u+v)}} \right] \\ &\qquad + \delta^v_\nu \frac{\beta^2}{4\pi} \left[\sum_{k\neq\ell} \frac{1}{[v(x_\ell,x_k)]_{-\operatorname{sgn}(u+v)}} - \sum_{k=1}^m \frac{1}{[v(x_\ell,y_k)]_{-\operatorname{sgn}(u+v)}} \right],\end{aligned} \tag{213}$$

where we recall that

$$[u(x_\ell,x_k)]_{-\operatorname{sgn}(u+v)} = \lim_{\epsilon\to 0} [u(x_\ell,x_k) - \mathrm{i}\epsilon \operatorname{sgn}((u+v)(x_\ell,x_k))]\,, \tag{214a}$$

$$[v(x_\ell, x_k)]_{-\operatorname{sgn}(u+v)} = \lim_{\epsilon \to 0} [v(x_\ell, x_k) - \mathrm{i}\epsilon \operatorname{sgn}((u+v)(x_\ell, x_k))] \tag{214b}$$

are the distributional boundary values obtained in the physical limit. We multiply by $f(x_\ell)$, sum over ℓ and rename summation indices to obtain

$$\begin{aligned}
&\sum_{k=1}^{m} f(x_k) \partial_\nu^{x_k} \ln\left(\left[\frac{\prod_{1\leq j<k\leq m} [u(x_j, x_k) v(x_j, x_k)]_- [u(y_j, y_k) v(y_j, y_k)]_-}{(-1)^m \prod_{j,k=1}^{m} [u(x_j, y_k) v(x_j, y_k)]_-} \right]^{\frac{\beta^2}{4\pi}} \right) \\
&\quad = \delta^u_\nu \frac{\beta^2}{4\pi} \left[\sum_{1\leq j<k\leq m}^{m} \frac{f(x_j) - f(x_k)}{[u(x_j, x_k)]_{-\operatorname{sgn}(u+v)}} - \sum_{j,k=1}^{m} \frac{f(x_j)}{[u(x_j, y_k)]_{-\operatorname{sgn}(u+v)}} \right] \\
&\qquad + \delta^v_\nu \frac{\beta^2}{4\pi} \left[\sum_{j=1}^{m} \sum_{k=j+1}^{m} \frac{f(x_j) - f(x_k)}{[v(x_j, x_k)]_{-\operatorname{sgn}(u+v)}} - \sum_{j,k=1}^{m} \frac{f(x_j)}{[v(x_j, y_k)]_{-\operatorname{sgn}(u+v)}} \right] \\
&\quad = \frac{\beta^2}{4\pi} \left[\sum_{1\leq j<k\leq m} [f(x_j) - f(x_k)] H_\nu(x_j, x_k) - \sum_{j,k=1}^{m} f(x_j) H_\nu(x_j, y_k) \right],
\end{aligned} \tag{215}$$

where in the last equality we used equation (140) in the limit $\epsilon \to 0$, which can be written as

$$H_u(x, y) = \frac{1}{[u(x, y)]_{-\operatorname{sgn}(u+v)}}, \quad H_v(x, y) = \frac{1}{[v(x, y)]_{-\operatorname{sgn}(u+v)}}. \tag{216}$$

By the same procedure, we also obtain the analogous equation with x and y exchanged. Similarly, we compute

$$\begin{aligned}
&\sum_{k=1}^{m} f(x_k) \partial_\nu^{x_k} \exp\left[-\frac{\beta^2}{2} \sum_{i,j=1}^{m} [W(x_i, x_j) - W(y_i, x_j) - W(x_i, y_j) + W(y_i, y_j)] \right] \\
&\quad = -\beta^2 \exp\left[-\frac{\beta^2}{2} \sum_{i,j=1}^{m} [W(x_i, x_j) - W(y_i, x_j) - W(x_i, y_j) + W(y_i, y_j)] \right] \\
&\qquad \times \sum_{j,k=1}^{m} f(x_k) \partial_\nu^{x_k} [W(x_k, x_j) - W(x_k, y_j)],
\end{aligned} \tag{217}$$

where the derivative acts on the first argument of W, as well as the analogous equation with x and y exchanged. It follows that the numerator of the Gell-

Mann–Low formula for the stress tensor, smeared with $\partial^\mu f$, reduces to

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int \cdots \int \sum_{\sigma_i = \pm 1} \omega^{0,0} \left(\mathcal{T} \left[T_{\mu\nu}(\partial^\mu f) \otimes \bigotimes_{j=1}^{n} V_{\sigma_j \beta}(x_j) \right] \right) \prod_{i=1}^{n} g(x_i) \, \mathrm{d}^2 x_i$$

$$= -\mathrm{i} \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \int \cdots \int \left[\frac{\prod_{1 \le j < k \le m} [u(x_j, x_k) v(x_j, x_k)]_- [u(y_j, y_k) v(y_j, y_k)]_-}{(-1)^m \prod_{j,k=1}^{m} [u(x_j, y_k) v(x_j, y_k)]_-} \right]^{\frac{\beta^2}{4\pi}}$$
$$\times \left[\sum_{i=1}^{m} [\partial_\nu f(x_i) + \partial_\nu f(y_i)] - \frac{\beta^2}{8\pi} \sum_{i=1}^{m} [\partial_\nu f(x_i) + \partial_\nu f(y_i)] \right.$$
$$\left. + \frac{\beta^2}{8\pi} \sum_{j=1}^{m} [\delta^u_\nu \partial_v f(x_j) + \delta^v_\nu \partial_u f(x_j) + \delta^u_\nu \partial_v f(y_j) + \delta^v_\nu \partial_u f(y_j)] \right]$$
$$\times \mu^{-m \frac{\beta^2}{2\pi}} \exp \left[-\frac{\beta^2}{2} \sum_{i,j=1}^{m} [W(x_i, x_j) - W(y_i, x_j) - W(x_i, y_j) + W(y_i, y_j)] \right]$$
$$\times \prod_{i=1}^{m} g(x_i) g(y_i) \, \mathrm{d}^2 x_i \, \mathrm{d}^2 y_i$$
$$+ \mathrm{i} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)!} \int \cdots \int \left[\frac{\prod_{1 \le j < k \le m+1} [u(x_j, x_k) v(x_j, x_k)]_- [u(y_j, y_k) v(y_j, y_k)]_-}{(-1)^{m+1} \prod_{j,k=1}^{m+1} [u(x_j, y_k) v(x_j, y_k)]_-} \right]^{\frac{\beta^2}{4\pi}}$$
$$\times \mu^{-(m+1) \frac{\beta^2}{2\pi}} \exp \left[-\frac{\beta^2}{2} \sum_{i,j=1}^{m+1} [W(x_i, x_j) - W(y_i, x_j) - W(x_i, y_j) + W(y_i, y_j)] \right]$$
$$\times [\partial_\nu f(x_{m+1}) + \partial_\nu f(y_{m+1})] \prod_{i=1}^{m+1} g(x_i) g(y_i) \, \mathrm{d}^2 x_i \, \mathrm{d}^2 y_i \,, \tag{218}$$

where we integrated some derivatives by parts, used that by assumption g is constant on the support of f, and for the last two terms in the stress tensor involving the vertex operators used the result (201) with z renamed to x_{m+1} and the analogous result for $V_{-\beta}$ with z renamed to y_{m+1}. Shifting in the last sum the summation index $m \to m-1$, and using that because of the symmetry of the integrand we can replace

$$\partial_\nu f(x_m) + \partial_\nu f(y_m) \to \frac{1}{m} \sum_{i=1}^{m} [\partial_\nu f(x_i) + \partial_\nu f(y_i)] \,, \tag{219}$$

the last sum in equation (218) coming from the vertex operators in the stress tensor cancels the first sum in brackets in the first sum in equation (218), completely analogous to the Euclidean case. However, the other terms do not cancel, and to remove them we need to use the finite renormalisation freedom we still have, redefining time-ordered products involving one $\mathcal{O}_{\mu\nu}$ and one vertex operator V_α.

Analogously to the redefinition (152) that was needed to renormalise the time-ordered products, we make a further redefinition of the form

$$\delta \mathcal{T}[\mathcal{O}_{\mu\nu}(z) \otimes V_\alpha(x)] = \left(c^u_\alpha \delta^u_\mu \delta^u_\nu + c^v_\alpha \delta^v_\mu \delta^v_\nu \right) \delta^2(z - x) \, \mathcal{T}[V_\alpha(x)] \tag{220}$$

with (finite) constants c^u_α and c^v_α; there is no need for a term proportional to $\eta_{\mu\nu}$ since it would cancel out in the stress tensor anyway. Analogously to the change (154), this induces a change in the expectation value of $\mathcal{O}_{\mu\nu}$ and thus $T_{\mu\nu}$ which reads

$$\begin{aligned}
\omega^{\Lambda,\epsilon}&\left(\delta\mathcal{T}\left[\bigotimes_{j=1}^n V_{\sigma_j\beta}(x_j) \otimes T_{\mu\nu}(z) \right] \right) = \omega^{\Lambda,\epsilon}\left(\delta\mathcal{T}\left[\bigotimes_{j=1}^n V_{\sigma_j\beta}(x_j) \otimes \mathcal{O}_{\mu\nu}(z) \right] \right) \\
&= \sum_{j=1}^n \left(c^u_{\sigma_j\beta} \delta^u_\mu \delta^u_\nu + c^v_{\sigma_j\beta} \delta^v_\mu \delta^v_\nu \right) \delta^2(z - x_j) \prod_{1 \leq i < j \leq n} \left[\mu^2 (-u_{ij} v_{ij} + \mathrm{i}\epsilon) \right]^{\sigma_i \sigma_j \frac{\beta^2}{4\pi}} \\
&\quad\times \exp\left[-\frac{\beta^2}{2} \sum_{i,j=1}^n \sigma_i \sigma_j W(x_i, x_j) \right] \left(\frac{\Lambda}{\mu} \right)^{\frac{\beta^2 \left(\sum_{j=1}^n \sigma_j \right)^2}{4\pi}} ,
\end{aligned} \tag{221}$$

since the redefinition (220) is traceless. In the physical limit $\Lambda, \epsilon \to 0$ again only neutral configurations with $\sum_{j=1}^n \sigma_j$ contribute, and summing over n and smearing with the adiabatic cutoff function g and the test function f, we obtain the change of the numerator of the Gell-Mann–Low formula for the stress tensor:

$$\begin{aligned}
&\sum_{n=0}^\infty \frac{\mathrm{i}^n}{n!} \int\cdots\int \sum_{\sigma_i = \pm 1} \omega^{0,0}\left(\delta\mathcal{T}\left[\bigotimes_{j=1}^n V_{\sigma_j\beta}(x_j) \otimes T_{\mu\nu}(f) \right] \right) \prod_{i=1}^n g(x_i) \,\mathrm{d}^2 x_i \\
&= \sum_{m=0}^\infty \frac{(-1)^m}{(m!)^2} \int\cdots\int \left[\frac{\prod_{1 \leq j < k \leq m} [u(x_j, x_k) v(x_j, x_k)]_- [u(y_j, y_k) v(y_j, y_k)]_-}{(-1)^m \prod_{j,k=1}^m [u(x_j, y_k) v(x_j, y_k)]_-} \right]^{\frac{\beta^2}{4\pi}} \\
&\quad\times \sum_{j=1}^m \left[\delta^u_\mu \delta^u_\nu \left(c^u_\beta f(x_j) + c^u_{-\beta} f(y_j) \right) + \delta^v_\mu \delta^v_\nu \left(c^v_\beta f(x_j) + c^v_{-\beta} f(y_j) \right) \right] \\
&\quad\times \mu^{-m \frac{\beta^2}{2\pi}} \exp\left[-\frac{\beta^2}{2} \sum_{i,j=1}^m \left[W(x_i, x_j) - W(y_i, x_j) - W(x_i, y_j) + W(y_i, y_j) \right] \right] \\
&\quad\times \prod_{i=1}^m g(x_i) g(y_i) \,\mathrm{d}^2 x_i \,\mathrm{d}^2 y_i ,
\end{aligned} \tag{222}$$

where we have used (as before) that only the terms with $n = 2m$ with m positive σ_j and m negative ones contribute, renamed the integration variables for $V_{-\beta}$ to y_i, and took into account that there are $\binom{n}{m} = (2m)!/(m!)^2$ possibilities to choose the m positive σ_j from a total of $n = 2m$ ones. Replacing $f \to \partial^\mu f$, the terms in the sum result in

$$\begin{aligned}
&\delta^u_\mu \delta^u_\nu \left(c^u_\beta \partial^\mu f(x_j) + c^u_{-\beta} \partial^\mu f(y_j) \right) + \delta^v_\mu \delta^v_\nu \left(c^v_\beta \partial^\mu f(x_j) + c^v_{-\beta} \partial^\mu f(y_j) \right) \\
&\quad= -2 \delta^u_\nu \left(c^u_\beta \partial_v f(x_j) + c^u_{-\beta} \partial_v f(y_j) \right) - 2 \delta^v_\nu \left(c^v_\beta \partial_u f(x_j) + c^v_{-\beta} \partial_u f(y_j) \right)
\end{aligned} \tag{223}$$

using the explicit form of the Minkowski metric in light cone coordinates (149), and adding this contribution to the numerator of the Gell-Mann–Low formula for the stress tensor, smeared with $\partial^\mu f$ (218), we can cancel the offending terms of the third sum in brackets by choosing

$$c^u_{\pm\beta} = c^v_{\pm\beta} = -\mathrm{i}\,\frac{\beta^2}{16\pi}\,. \tag{224}$$

In effect, the result of this redefinition is to remove the local terms in the renormalised $[H_\mu(x,y)H_\nu(x,y)]^{\mathrm{ren}}$ (151) for the uu and vv components, such that $[H_u(x,y)H_u(x,y)]^{\mathrm{ren}} = 4\pi\mathrm{i}\,\partial_u^2 H^{\mathrm{F}}(x,y)$ and analogously for the vv component.

The remaining term in equation (218) is of the same form as the contribution of $V_{\pm\beta}$ (201), and as in the Euclidean case it follows that a modified stress tensor is conserved in the quantum theory: we have

$$\sum_{n=0}^{\infty}\frac{1}{n!}\int\cdots\int\sum_{\sigma_i=\pm1}\omega^{0,0}\left(\mathcal{T}\left[\hat{T}_{\mu\nu}(\partial^\mu f)\otimes\bigotimes_{j=1}^{n}V_{\sigma_j\beta}(x_j)\right]\right)\prod_{i=1}^{n}g(x_i)\,\mathrm{d}^2x_i = 0 \tag{225}$$

with (15)

$$\hat{T}_{\mu\nu} \equiv T_{\mu\nu} - \frac{\beta^2}{8\pi}g\eta_{\mu\nu}(V_\beta + V_{-\beta})\,. \tag{226}$$

One might ask if a further redefinition of time-ordered products could be used to get rid of the extra term in equation (226), such that the classical stress tensor would also be conserved in the quantum theory. However, this is impossible since the term in question is proportional to $\eta_{\mu\nu}$, and modifying $\mathcal{O}_{\mu\nu}$ by any such term has no effect on the stress tensor. Moreover, redefinitions of time-ordered products only involving V_β are not allowed by dimensional reasons in the finite regime $\beta^2 < 4\pi$, so the modified stress tensor (226) is unique. □

Data availability statement. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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