The Borchers-Uhlmann Algebra

and its Descendants

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Topics:

1. The BU algebra
2. Wightman functionals
3. Abundance of states satisfying parts of the W-axioms
4. Bounded Bose fields
5. Noncommutative moment problems
6. Deformed BU algebras
7. An extension problem for states

The seminal papers:

A.S. Wightman: *Quantum Field Theory in Terms of Vacuum Expectation Values*, Phys. Rev. **101**, 860– 866 (1956)

H.J. Borchers: *On Structure of the Algebra of Field Operators*, Nuovo Cimento, **24**, 1418–1440 (1962)

A. Uhlmann: *Über die Definition der Quantenfelder nach Wightman und Haag*, Wiss. Zeitschr. Karl-Marx Univ. Leipzig, **11**, 213–217 (1962)

H.J. Borchers: *Algebraic aspects of Wightman field theory*. In: Sen, R.N., Weil, C. (Eds.): Statistical mechanics and field theory, Haifa Lectures 1971

The *Borchers-Uhlmann (BU) Algebra* is the tensor algebra over the space of test functions for Wightman quantum fields. For concreteness we consider here the case of a scalar, hermitian, Bose field. The test function space is then Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$, $d \geq 2$, with its usual topology, and the BU Algebra is

$$\underline{\mathcal{S}} = \bigoplus_{n=0}^{\infty} \mathcal{S}_n$$

with

$$\mathcal{S}_0 = \mathbb{C}, \quad \mathcal{S}_n = \mathcal{S}(\mathbb{R}^{nd}).$$

Its elements are sequences

$$\underline{f} = (f_0, ..., f_N, 0, ...), \quad f_n \in \mathcal{S}_n, \quad N < \infty$$

and the product is the tensor product:

$$(\underline{f} \otimes \underline{g})_n(x_1, \ldots, x_n) = \sum_{\nu=0}^{n} f_\nu(x_1, ..., x_\nu) g_{n-\nu}(x_{\nu+1}, ..., x_n)$$

There is also an antilinear involution:

$$(\underline{f}^*)_n(x_1, ..., x_n) = \overline{f_n(x_n, ..., x_1)}.$$

Via GNS construction there is a one-to-one correspondence between cyclic *-representations of $\underline{\mathcal{S}}$ by (in general unbounded) Hilbert space operators $\phi(\underline{f})$ and positive linear functionals on $\underline{\mathcal{S}}$, i.e.,

$$\omega = (\omega_0, \omega_1, ...) \in \underline{\mathcal{S}}', \quad \omega_n \in \mathcal{S}'_n$$

satisfying

$$\omega(\underline{f}^* \otimes \underline{f}) \geq 0 \text{ for all } \underline{f}.$$

The correspondence is given by

$$\omega(\underline{f}) \equiv \sum_n \omega_n(f_n) = \langle \Omega, \phi(\underline{f})\Omega \rangle$$

with Ω the cyclic vector. Moreover, $\|\Omega\| = 1$ iff $\omega_0 = 1$.

We denote the set of (continuous) positive linear functionals by $\underline{\mathcal{S}}'^{+}$; its normalized elements are, as usual, called *states*.

One of the first general mathematical results on the BU algebra is due to G. Laßner and A. Uhlmann (1968): The positive linear functionals separate points, so $\underline{\mathcal{S}}$ has a faithful Hilbert space representation. A stronger result is the following (JY, 1973):

Theorem 1 (Existence of 'large' positive functionals) *For every continuous seminorm p on $\underline{\mathcal{S}}$ there is an $\omega \in \underline{\mathcal{S}}'^{+}$ such that*

$$p(\underline{f})^2 \leq \omega(\underline{f}^* \otimes \underline{f}).$$

Corollary 1 *A linear functional $T \in \underline{\mathcal{S}}'$ belongs to the linear span of the states iff, for some continuous seminorm p on $\underline{\mathcal{S}}$,*

$$|T(\underline{f} \otimes \underline{g})| \leq p(\underline{f})p(\underline{g}) \quad \text{for all } \underline{f} \text{ and } \underline{g}.$$

Example of a $T \in \underline{\mathcal{S}}'$ that can *not* be written as a linear combination of positive functionals:

$$T_n = \delta^{(n)} \otimes \cdots \otimes \delta^{(n)}$$

The positivity condition encodes only one physical aspect of QFT, the *probability interpretation*. No less important are *Causality*, *Stability* and *Relativistic Invariance* that are taken into account by considering some *ideals* and *automorphisms* of $\underline{\mathcal{S}}$:

Locality ideal $\mathcal{I}_{\mathrm{C}}$:

Two-sided ideal generated by $f \otimes g - g \otimes f$,

supports of f and g space-like separated.

Spectrum ideal $\mathcal{L}_{\mathrm{sp}}$:

Left ideal, generated by elements of the form $\int \alpha_a \underline{f}\, h(a) da$ with support of $\tilde{h}$ in the complement of the forward light cone. Here α_a is translation by $a \in \mathbb{R}^d$:

$$(\alpha_a \underline{f})_n(x_1, ..., x_n) = f_n(x_1 - a, ..., x_n - a).$$

More generally, a *Poincaré transformation* $(a, \Lambda) \in \mathcal{P}_+^\uparrow$ operates on $\underline{\mathcal{S}}$ by automorphisms as

$$(\alpha_{(a,\Lambda)} \underline{f})_n(x_1, ..., x_n) = f_n(\Lambda^{-1}(x_1 - a), ..., \Lambda^{-1}(x_n - a)).$$

A *Wightman functional* is, by definition, a (normalized) linear functional ω on the BU algebra satisfying the following conditions (*'W-conditions'*):

- $\omega \in \underline{\mathcal{S}}'^{+}$ (positivity)

- $\omega(\mathcal{I}_{\mathsf{c}}) = \{0\}$ (locality)

- $\omega(\mathcal{L}_{\mathsf{sp}}) = \{0\}$ (spectrum condition)

- $\omega \circ \alpha_{(a,\Lambda)} = \omega$ for all (a, Λ) (invariance)

Wightman functionals are, via GNS construction, in a one-to-one correspondence with Wightman quantum fields. Hence the interest in this particular subset of $\underline{\mathcal{S}}'^{+}$, denoted by $\mathcal{W}$.

Elementary operations on $\mathcal{W}$:

- Convex combinations

- p-product: $\left(\omega^{(1)} \mathsf{p}\, \omega^{(2)}\right)_n (x_1, ..., x_n) = \omega_n^{(1)}(x_1, ..., x_n)\omega_n^{(2)}(x_1, ..., x_n)$

- s-product: $\left(\omega^{(1)} \mathsf{s}\, \omega^{(2)}\right)_n (x_1, ..., x_n)$
$$= \sum\nolimits_{\text{part}} \omega_\mu^{(1)}(x_{i_1}, ..., x_{i_\mu})\omega_\nu^{(2)}(x_{j_1}, ..., x_{j_\nu})$$

Remark: The s-product can be defined for arbitrary functionals in $\underline{\mathcal{S}}'$. Also infinite power series of T w.r.t. the s-product can be defined, in particular $\exp|_{\mathsf{s}} T$, for $T \in \underline{\mathcal{S}}'$. If T satisfies the linear W-conditions then so does $\exp|_{\mathsf{s}} T$. If $T_0 \neq 0$ one can conversely define $\log|_{\mathsf{s}} T \equiv T^t$. This is just the *truncated functional* corresponding to T.

A *sufficient* (but no means necessary) condition for $\exp|_{\mathsf{s}} T$ to be positive is that T is *conditionally positive*, i.e., that $T(\underline{f}^* \otimes \underline{f}) \geq 0$ for all $\underline{f}$ with $f_0 = 0$.

A study of prime decompositions with respect to the the s-product was undertaken by G. Hegerfeldt (1975; 1985).

The known examples of W-fields in $d \geq 4$ are

a) Generalized free fields

b) Generalized Wick-polynomials in such fields

c) Fields obtained from a) and b) by convex combinations and by s- and p-products.

Why is the construction of W-functionals so difficult?

Comparison with the C^* situation:

For abundance of states one may rely on *extension theorems*. Essential point: Cone of positive elements has *interior points*.

To obtain translationally invariant states one can use that the translation group is amenable, i.e, has an *invariant mean*.

For states vanishing on a left ideal: Every left ideal is generated by a *projector* E in the enveloping W^* algebra. For any state ω, the projected state $A \mapsto \omega((1-E)A(1-E))$ vanishes on the ideal.

None of these tools are *a priori* available for the BU algebra.

Nevertheless, the algebraic formulation of Wightman theory raises natural questions about the compatibility between positivity and one or more of the linear W-conditions. Some of these questions could be answered in the 80's and 90's. Examples:

Theorem 2 (Spectrum condition and translational invariance) *For every continuous, translationally invariant seminorm p on $\underline{\mathcal{S}}$ vanishing on $\mathcal{L}_{\mathsf{sp}}$ there is a translationally invariant, positive linear functional ω, vanishing on $\mathcal{L}_{\mathsf{sp}}$, with*

$$p(\underline{f})^2 \leq \omega(\underline{f}^* \otimes \underline{f}).$$

Corollary 2 *A linear functional $T \in \underline{\mathcal{S}}'$ belongs to the linear span of the translationally invariant states annihilating $\mathcal{L}_{\mathsf{sp}}$ if and only if, for some continuous seminorm p on $\underline{\mathcal{S}}$,*

$$|\int T(\underline{f} \otimes \alpha_a \underline{g}) h(a) da| \leq p(\underline{f}) p(\underline{g}) \sup_{q \in \mathsf{V}^+} |\tilde{h}(q)| \text{ for all } \underline{f}, \underline{g} \in \underline{\mathcal{S}},\ h \in \mathcal{S}_1.$$

Theorem 3 (Bounded representations and Locality) *The algebra $\underline{\mathcal{S}}/\mathcal{I}_{\mathrm{C}}$ has a faithful, translationally covariant Hilbert space representation by bounded operators.*

This last result is rather surprising and leads to the question whether the field operators of a Bose field can be bounded operators in Wightman theory, i.e., whether there exist bounded representations of $\underline{\mathcal{S}}/\mathcal{I}_{\mathrm{C}}$ satisfying the spectrum condition. Such fields would certainly be new examples in $d \geq 3$!

In $d = 2$ the first example was given by Buchholz (1994?) and subsequently it was shown by Rehren (1996) that bounded Bose fields are, in fact, quite abundant in $d = 2$. A general structure analysis of such fields satisfying Huygens' principle was undertaken by Baumann, and by Grott and Rehren (2000), leading to the conclusion that in this case there is no scattering, even if the W-function can look quite complicated.

Noncommutative moment problems (Borchers, JY, 1992)

Denote the 'partially symmetric' tensor algebra $\underline{\mathcal{S}}/\mathcal{I}_{\mathrm{C}}$ by $\mathfrak{P}$. It can be regarded as a free *-algebra with generators in $\mathcal{S}_h$ (real part of $\mathcal{S}$) and relations

$$fg = gf \quad \text{if } f \text{ and } g \text{ have space-like separated supports.}$$

The unit is denoted by $\mathbf{1}$. We now extend $\mathfrak{P}$ to an algebra $\mathfrak{F}$ by adding new generators, denoted by $R_+(f)$ and $R_-(f) = R_+(f)^*$ for $f \in \mathcal{S}_h$, and the relations

$$\begin{aligned}(f + i\mathbf{1})R_+(f) &= R_+(f)(f + i\mathbf{1}) = \mathbf{1} \\ (f - i\mathbf{1})R_-(f) &= R_-(f)(f - i\mathbf{1}) = \mathbf{1} \\ R_\pm(f)R_\pm(g) &= R_\pm(g)R_\pm(f) \quad \text{if supp } f \text{ and supp } g \text{ space-like.}\end{aligned}$$

The $R_{\pm}(f)$ can thus be regarded as abstract resolvents of the generators f and are also denoted by $(f+i\mathbf{1})^{-1}$.

(Remark: Such abstract resolvents have recently been applied by Buchholz and Grundling (2008) to define a novel C^* algebra for the CCR.)

The set of bounded elements of $\mathfrak{F}$ is defined as the algebra $\mathfrak{B}$ generated by elements of the form $f^n(f^2+1)^{-m}$ with $n \leq 2m$. Finally we define a 'positive cone' in $\mathfrak{F}$ as

$$\mathfrak{F}^+ = \left\{ \sum_{ijk} P^*_{ik} B^*_{jk} B_{jk} P_{ik} \,|\, B_{jk} \in \mathfrak{B}, P_{ik} \in \mathfrak{P} \right\}$$

Theorem 4 (Self-adjoint extensions) *Let* ω *be a state on* $\mathfrak{P}$ *with corresponding field operators* $\phi(f)$ *on a Hilbert space* $\mathcal{H}$ *for* $f \in \mathcal{S}_h$. *The following are equivalent:*

(i) ω *is positive on* $\mathfrak{P} \cap \mathfrak{F}^+$.

(ii) The operators $\phi(f)$ *have self-adjoint extensions* $\widehat{\phi(f)}$ *on a Hilbert space* $\widehat{\mathcal{H}} \supset \mathcal{H}$ *such that* $\widehat{\phi(f)}$ *and* $\widehat{\phi(g)}$ *commute strongly (i.e., bounded functions of the operators commute) if supp* f *and supp* g *space-like separated.*

This general result can be combined with some earlier ideas of Powers (1974), and of Driessler, Summers and Wichmann (1986) to derive a criterion for the existence of a local net of von Neumann algebras associated with a given Wightman field.

Two more concepts are needed:

1. The *weak commutant* of a set $\mathcal{M}$ of closeable, but in general unbounded, operators with dense domain $\mathcal{D}$ a Hilbert space $\mathcal{H}$:

$$\mathcal{M}^w = \{C \in B(\mathcal{H}) \,|\, \langle C\psi, A\varphi\rangle = \langle A^*\psi, C^*\varphi\rangle \text{ for all } A \in \mathcal{M},\ \psi,\varphi \in \mathcal{D}\}\,.$$

2. Let $\mathfrak{A}$ be a *-algebra and $\mathfrak{A}^+$ the cone generated by squares A^*A, $A \in \mathfrak{A}$. A state ω on $\mathfrak{A}$ is called *centrally positive* with respect to an element $A_0 \in \mathfrak{A}$ if ω is positive on all elements of the form $\sum A_0^n A_n$ with $A_n \in \mathfrak{A}$ such that $\sum \lambda^n A_n \in \mathfrak{A}^+$ for all $\lambda \in \mathbb{R}$.

Theorem 5 (Local nets of von Neumann algebras) *Let ω be a Wightman state, let $\mathfrak{P}(\mathcal{O})$ be the subalgebra of $\mathfrak{P}$ generated by the test functions with support in $\mathcal{O}$ for an open set $\mathcal{O} \subset \mathbb{R}^d$ and let $\mathcal{P}(\mathcal{O})$ be the corresponding *-algebra of (in general unbounded) operators in the GNS representation defined by ω. If the weak commutants $\mathcal{P}(\mathcal{O})^w$ are algebras, then the following statements are equivalent:*

(i) The net of von Neumann algebras $\mathcal{A}(\mathcal{O}) = \mathcal{P}(\mathcal{O})^{w\prime}$ is a local net, i.e., algebras corresponding to space like separated regions commute.

(ii) For every open set $\mathcal{O}$ and every real test function f with support in $\mathcal{O}$ the state ω is centrally positive w.r.t. f on the subalgebra of $\mathfrak{P}$ generated by f and test functions with support in the causal complement $\mathcal{O}^c$ of $\mathcal{O}$.

Remark: If $\mathcal{P}(\mathcal{O})^w$ is an algebra, then $\mathcal{A}(\mathcal{O}) = \mathcal{P}(\mathcal{O})^{w\prime}$ is the minimal von Neumann algebra to which the field operators $\phi(f)$, supp $f \subset \mathcal{O}$, are affiliated.

A sufficient criterion for $\mathcal{P}(\mathcal{O})^w$ to be an algebra can be formulated in terms of *generalized H-bounds*:

We say that a Wightman field satisfies generalized H-bounds of order $\alpha > 0$, if $\phi(f)^{**} \exp(-(1 + H^2)^{\alpha/2})$ is a bounded operator for all f. Here H is the Hamiltonian.

The following was proved by Driessler, Summers and Wichmann (1986):

Theorem 6 *If a Wightman field satisfies generalized H-bounds of order $\alpha < 1$, then the weak commutants $\mathcal{P}(\mathcal{O})^w$ are von Neumann algebras.*

Deformations of the BU algebra

In the past few years there has been much interest in QFT on noncommutative space-times ('noncommutative QFT'). These developments have also led to interesting results about locality preserving deformations of algebras corresponding to space-like wedges in Minkowski space. (Grosse and Lechner (2007– 2008), Buchholz and Summers (2008), Morfa-Morales (2009).)

A formulation employing a deformation of the BU algebra rather than von Neumann algebras was recently presented by Grosse and Lechner. Here a brief description of this deformation will be given.

It is based on the Moyal tensor product of two functions $f \in \mathcal{S}(\mathbb{R}^{4n})$, $g \in \mathcal{S}(\mathbb{R}^{4m})$defined by an antisymmetric 4×4 matrix θ:

$$(f \otimes_\theta g)(x_1, \ldots, x_{n+m}) =$$
$$\pi^{-4} \int d^4\xi \int d^4q \, \mathrm{exp}(-2i\xi{\cdot}q) f(x_1{-}\xi, ..., x_n{-}\xi) g(x_{n+1}{-}\theta q, ..., x_{n+m}{-}\theta q)$$

This extends to a bilinear, separately continuous and associative product

$$\otimes_\theta : \underline{\mathcal{S}} \times \underline{\mathcal{S}} \to \underline{\mathcal{S}}.$$

Note, however, that in general

$$(\underline{f} \otimes_\theta \underline{g}) \otimes_{\theta'} \underline{h} \neq \underline{f} \otimes_\theta (\underline{g} \otimes_{\theta'} \underline{h}) \quad \text{if } \theta \neq \theta'.$$

For a given Wightman state ω a deformed state is defined by

$$\omega^{\theta}(f_1 \otimes \cdots \otimes f_n) = \omega(f_1 \otimes_{\theta} \cdots \otimes_{\theta} f_n).$$

In terms of the Fourier transformations of the n-point functions, $\tilde{\omega}_n$, this is equivalent to

$$\tilde{\omega}_n^{\theta}(p_1, ..., p_n) = \prod_{1 \leq l < r \leq n} \exp(-\tfrac{i}{2} p_l \cdot \theta p_r) \tilde{\omega}_n(p_1, ..., p_n).$$

Note that, since $\tilde{\omega}_2(p_1, p_2)$ is supported on $p_1 = -p_2$ by translational invariance, the twisting factor $\prod_{1 \leq l < r \leq n} \exp(-\frac{i}{2} p_l \cdot \theta p_r)$ introduces a non-trivial θ-dependence only in the higher n-point functions, $n \geq 3$.

An important remark is that the different deformations of a given Wightman field corresponding to different θ's can be defined on a common Hilbert space, namely the Hilbert space of the undeformed field, defined by GNS construction from ω.

Moreover, there is a correspondence between θ and a space-like wedge region $W(\theta)$ in Minkowski space; after normalizing the (purely imaginary) eigenvalues of θ this correspondence is one-to-one. A key result of Grosse and Lechner is

Theorem 7 (Wedge localization of deformed fields) *If ϕ is a Wightman field then ϕ^θ is wedge localized in the sense that $\phi^\theta(x)$ and $\phi^{\theta'}(y)$ commute if $x + W(\theta)$ and $y + W(\theta')$ are space like separated.*

An extension problem for states

The following is well known: Any two-point function satisfying the restriction of the W-conditions to $\mathcal{S}_2$ can be extended to a Wigthman functional, namely that of a generalized free field with the given two-point function. Moreover, if the Lehmann-weight decreases sufficiently fast with the mass (in particular if the KG equation is fulfilled for the 2-point function) the extension is unique.

A natural question is the following: Given n-point functions $\omega_2, ..., \omega_{2m}$ with $m \geq 2$ that satisfy the W-conditions restricted to $\sum_{n=0}^{2m} \mathcal{S}_n$, which additional conditions are necessary in order that this finite sequence can be extended to a Wightman functional on $\underline{\mathcal{S}}$?

This is a formidably difficult problem, already for $m = 2$! It is instructive to compare it with a different, but closely related question, the *representation problem* for reduced density matrices in nonrelativistic many-body quantum mechanics.

The Hamiltonian of a system of particles in an external potential V and with a two-body interaction potential v can, in second quantized notation, be written

$$H = \int dx \, \{-a_x^* \Delta a_x + V(x) a_x^* a_x\} + \int \int dx dy \, v(x-y) a_x^* a_y^* a_y a_x$$

Given an N particle wave function and denoting by $\langle\cdot\rangle$ the corresponding expectation value one sees that the expectation value $\langle H\rangle$ is completely determined by the *reduced one- and two-particle density matrices*

$$\rho^{(1)}(x;x') = \langle a_x^* a_{x'}\rangle \qquad \text{and} \qquad \rho^{(2)}(x,y;y',x') = \langle a_x^* a_y^* a_{y'} a_{x'}\rangle.$$

(In fact, since $\rho^{(1)}(x;x') = \int dy\, \rho^{(2)}(x,y;y,x')$, only $\rho^{(2)}$ is needed.)

This observation (Coleman, 1951) leads, at first sight, to a vast simplification of the question of ground state energies of many-body systems: One has only to take the infimum over all possible $\rho^{(2)}$!

But there is a big snag: It is not known which functions on $\mathbb{R}^{4d}$ are possible reduced 2-body density matrices of many-body states. Some *necessary* conditions are known, however, in particular for Fermions. Taking the infimum over $\rho^{(2)}$'s satisfying such conditions leads in any case to lower bounds to ground states energies. Such bounds can even in some cases be quite good numerically (Cancès, Stoltz and Lewin (2006)).

There is an analogy with the extension problem for Wightman functions, because $\rho^{(1)}$ and $\rho^{(2)}$ are, respectively, two-point and four-point functions on the CCR or CAR algebra. A necessary condition is that these functions can be complemented to obtain a positive linear functional on this algebra.

In the Wightman case very little is known about this extension problem. A simple *necessary* condition (for the case $m = 2$, i.e, the 4-point function) is the following consequence of the Cauchy-Schwarz inequality:

$$|\omega_4(f_3 \otimes f_1)| \leq p(f_3)\, \omega_2(f_1^* \otimes f_1)^{1/2}$$

for all $f_3 \in \mathcal{S}_3$, $f_1 \in \mathcal{S}_1$, with a continuous seminorm p. The linear W-conditions imply further conditions on p, and since p really comes from a positive definite 6-point function, the Cauchy-Schwarz inequality can be iterated, implying further restrictions.

If the linear conditions are ignored, i.e, if the question is just considered for functionals on $\underline{\mathcal{S}}$, there are examples that show that infinitely many conditions are, indeed, needed. Namely, the following can be shown:

Given any N, there are $\omega_2 \in \mathcal{S}_2'$ and $\omega_4 \in \mathcal{S}_4'$ such that there exists an extension of these functions to a functional ω on $\underline{\mathcal{S}}$ with

$$\omega(\underline{f}^* \otimes \underline{f}) \geq 0 \quad \text{for all } \underline{f} \in \bigoplus_{n=0}^{N} \mathcal{S}_n$$

but there exists *no* extension with $\omega(f_{N+1}^* \otimes f_{N+1}) \geq 0$ for all $f_{N+1} \in \mathcal{S}_{N+1}$.

Even if a complete solution to the extension problem is probably not feasible, the search for new necessary conditions might still merit further effort.