Local Conformal Nets and Vertex Operators

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Introduction

- The study of conformal field theory (CFT) in two space-time dimensions has found applications to different areas of physics and mathematics such as string theory, critical phenomena, infinite dimensional Lie algebras, number theory, finite simple groups, 3-manifold invariants, the theory of subfactors and noncommutative geometry.
- Vertex operator algebras (VOAs) and conformal nets on S^1 give two different mathematically rigorous frameworks for chiral conformal quantum field theories (chiral CFTs) .
- From the mathematical point of view these two framework looks quite different but they show their common physics root many structural similarities.
- In this talk I will report on some recent results and work in progress on the general connection of these two axiomatizations of chiral CFT.

Chiral CFT

- Two-dimensional CFT $\equiv$ scaling invariant quantum field theories on the two-dimensional Minkowski space-time admitting conformal symmetry. Certain relevant fields (the chiral fields) depend only on $x - t$ (right-moving fields) or on $x + t$ (left-moving fields).
- Chiral CFT $\equiv$ CFT generated by left-moving (or right-moving) fields only. Chiral CFTs can be considered as QFTs on $\mathbb{R}$ and by conformal symmetry on its compactification $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Hence we can consider quantum fields on the unit circle $\Phi(z)$, $z \in S^1$ and the corresponding smeared field operators $\Phi(f)$, $f \in C^\infty(S^1)$.
- Let $\mathcal{F}$ be a two-dimensional CFT and let $\mathcal{F}_+$ and $\mathcal{F}_-$ be the subtheories generated by the left-moving and right-moving fields respectively. Then, these chiral subtheories decouple and we have an embedding $\mathcal{F}_+ \otimes \mathcal{F}_- \subset \mathcal{F}$.
- Typically the latter embedding is proper but one can try to reconstruct the possible full 2D theories from the chiral subtheories. In this sense the chiral CFTs can be considered as the building blocks of two-dimensional CFT.

Conformal nets on S^1

- Conformal nets are the chiral CFT version of algebraic quantum field theory (AQFT).
- In the conformal net approach to CFT the theory is formulated in terms of von Neumann algebras namely algebras of bounded operators on a Hilbert space containing the identity and closed under taking adjoints and weak limits.
- A (local) conformal net $\mathcal{A}$ on $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ an inclusion preserving map $I \mapsto \mathcal{A}(I)$ from the set of (proper) intervals of S^1 into the set of von Neumann algebras acting on a fixed Hilbert space $\mathcal{H}_{\mathcal{A}}$ (the vacuum sector).
- The map is assumed to satisfy certain natural (and physically motivated) conditions: locality; conformal covariance; energy bounded from below; existence of the vacuum $\Omega \in \mathcal{H}_{\mathcal{A}}$.
- Conformal covariance is formulated through the existence of a continuous projective unitary representation of $\mathrm{Diff}^+(S^1)$.
- Locality means that $[\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\}$ whenever $I_1 \cap I_2 = \varnothing$

Vertex operator algebras

- In the vertex operator algebra approach to CFT the theory is formulated in terms of fields i.e. operator valued formal distributions (equivalently formal power series with operator coefficients) with some additional requirements.
- A vertex operator algebra (VOA) is a vector space V (the vacuum sector) together with a linear map (the state-field correspondence)

$$a \mapsto Y(a,z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}, \quad a_{(n)} \in \mathrm{End}(V)$$

  from V into the set of fields acting on V.
- The map $a \mapsto Y(a,z)$ is assumed to satisfy certain natural (and physically motivated) conditions: locality; conformal covariance; energy bounded from below; vacuum.
- Conformal covariance is formulated at the infinitesimal level through the existence of a representation of the Virasoro algebra.
- Locality means that, for any pair $a, b \in V$ there is an integer $N \geq 0$ such that $(z-w)^N [Y(a,z), Y(b,w)] = 0$.

- The fields $Y(a,z)$ are called vertex operators. The family of vertex operators should be considered as the family of "all" quantum fields of the theory in contrast with other approaches where one considers only a suitable family of generators.
- The notion of VOA is a special important case of the notion of vertex algebra.
- In order to make contact with the theory of conformal nets we need a unitary structure on $V \Rightarrow$ unitary VOAs. In this case the uniqueness of the vacuum for conformal nets (irreducibility) corresponds to the assumption that V is a simple VOA.
- If $a \in V$ is homogeneous of conformal weight $d \in \mathbb{Z}_{\geq 0}$, i.e. if $L_0 a = da$, where L_0 is the conformal energy operator, it is useful to introduce the notation $a_n := a_{(n+d-1)}$ so that $Y(a,z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-d}$. If $a \in V$ is not homogeneous then it can be written as a linear combination of homogenous vectors and a_n is defined by linearity.
- From now on V will be a simple unitary VOA.

From VOAs to conformal nets

- The general problem of constructing conformal nets from VOAs has been recently considered by S. C., Y. Kawahigashi, R. Longo and M. Weiner: arXiv:1503.01260 [math.OA] (to appear in Memoirs of the AMS), [CKLW2015].
- We assume that V is energy-bounded i.e. that for every $a \in V$ there exist positive integers s, j and a constant $K > 0$ such that

$$\|a_n b\| \leq K(|n|+1)^s \|(L_0 + 1_V)^j b\| \ \forall n \in \mathbb{Z}, \ \forall b \in V.$$

- Let $\mathcal{H}_V$ be the Hilbert space completion of V and let $f \in C^\infty(S^1)$ with Fourier coefficients $\hat{f}_n$. For every $a \in V$ we define the operator $Y^0(a, f)$ on $\mathcal{H}_V$ with domain V by

$$Y^0(a,f)b = \sum_{n \in \mathbb{Z}} a_n \hat{f}_n b \quad \text{for } b \in V.$$

It is a closable operator and we denote its closure by $Y(a, f)$ (smeared vertex operator).

- If $L_0 a = d_a a$ then we can use the formal notation

$$Y(a,f) = \oint_{S^1} Y(a,z) f(z) z^{d_a} \frac{\mathrm{d}z}{2\pi i z}.$$

- We define a map $\mathcal{A}_V$ from the set of intervals of S^1 into the the set of von Neumann algebras on $\mathcal{H}_V$ by

$$\mathcal{A}_V(I) = \text{von Neumann algebra generated by}$$

$$\{Y(a,f) : a \in V, f \in C_c^\infty(I)\}.$$

- Note that in the definition of $\mathcal{A}_V$ all fields are considered, not only a family of generators.
- It is clear that the map $I \mapsto \mathcal{A}_V(I)$ is inclusion preserving.
- Definition [CKLW2015]: V is strongly local if $\mathcal{A}_V$ satisfies locality.
- Strong locality does not follow in a obvious way from VOA locality because the von Neumann algebras generated by two unbounded operators commuting on a common invariant core need not to commute in general (Nelson's examples).

For a strongly local V we have the following results [CKLW2015]:

- $\mathcal{A}_V$ is a conformal net on S^1.
- Different unitary structures on V give rise to isomorphic (unitarily equivalent) conformal nets.
- The map $V \mapsto \mathcal{A}_V$ is "well behaved". Natural constructions in the VOA setting (subVOAs, tensor products) preserve strong locality.
- Many examples of unitary VOAs are known to be strongly local: unitary VOAs generated affine Lie algebras, the corresponding coset and orbifold subalgebras; unitary Virasoro VOAs; unitary VOAs with central charge $c = 1$; the moonshine VOA $V^\natural$ whose automorphism group is the monster group $\mathbb{M}$, the even shorter moonshine VOA $VB^\natural_{(0)}$ whose automorphism group is the baby monster group $\mathbb{B}$.

Back to VOAs, two conjectures and representation theory

- In 1996 K. Fredenhagen and M. Jörss proposed a construction of certain fields staring form a conformal net $\mathcal{A}$ (FJ fields).
- In our work we show that if V is strongly local then the FJ fields of $\mathcal{A}_V$ give back the vertex operators of V.
- We also show that if $\mathcal{A}$ is conformal net whose FJ fields satisfy appropriate energy bounds then there is a strongly local VOA V such that $\mathcal{A} = \mathcal{A}_V$.
- Conjecture 1. [CKLW2015] Every simple unitary VOA is strongly local.
- Conjecture 2. [CKLW2015] For every conformal net $\mathcal{A}$ there is a strongly local VOA V such that $\mathcal{A} = \mathcal{A}_V$.
- There is a encouraging ongoing work (S.C. and L. Tomassini) on Conjecture 2. This work could also give many other examples of strongly local VOAs such as unitary lattice VOAs, unitary VOAs with $c < 1$, unitary framed VOAs ... and hence also some further evidence on the validity of Conjecture 1.

- Conformal nets and VOAs have very interesting representation theories (theory of superselection sectors).
- These representation theories play a crucial role in the reconstruction problem of full CFTs from chiral subtheories.
- These representation theories are also very important for the construction and classification of chiral CFTs. For this reason the study of the above conjectures should also requires a direct connection between the representation theories VOAs and those of the corresponding of conformal nets.
- Connecting the representation theories in a direct way is interesting in itself and has many potential applications. Some recent progress in this direction have been made by S.C, M. Weiner and F. Xu [CWX$\geq$2017] (in preparation).

Representations of conformal nets

- Let $\mathcal{A}$ be a conformal net on S^1. A representation π of $\mathcal{A}$ is a family $\{\pi_I : I \subset S^1 \text{ is an interval}\}$, where each π_I is a representation of $\mathcal{A}(I)$ on a fixed Hilbert space $\mathcal{H}_\pi$, which is compatible with the net structure, i.e. $\pi_{I_2} \restriction_{\mathcal{A}(I_1)} = \pi_{I_1}$ if $I_1 \subset I_2$.
- In this talk any representation π will be locally normal i.e. such that π_I is normal for every interval $I \subset S^1$. A representation π is automatically locally normal if $\mathcal{H}_\pi$ is separable.
- The concepts of direct sums, subrepresentations, irreducibility .... for representations of conformal nets can be defined in a natural way.

Unitary VOA modules

- Let V be a simple unitary VOA. A vertex algebra module for V is a vector space M together with a linear map $a \mapsto Y_M(a,z) = \sum_{n\in\mathbb{Z}} a^M_{(n)} z^{-n-1}$ which is compatible with the vertex algebra structure of V i.e. it satisfies the so called Borcherds identity and moreover, $Y_M(\Omega, z) = 1_M$, where $\Omega \in V$ is the vacuum vector.
- If M is a vertex algebra module for V then M carries a representation of the Virasoro algebra. In particular there is on M a conformal energy operator L_0^M and we denote by M_h the eigenspaces $\mathrm{Ker}\left(L_0^M - h1_M\right)$, $h \in \mathbb{C}$.
- A VOA module for V is a vertex algebra module M such that $M = \bigoplus_{h\in\mathbb{C}} M_h$, with $M_h = \{0\}$ if the real part of h is sufficiently negative and $\dim(M_h) < \infty$ for all $h \in \mathbb{C}$.

- A unitary vertex algebra module for V is a vertex algebra module M with scalar product $(\cdot|\cdot)_M$ which is compatible with the unitary structure of V. In this case the representation of the Virasoro algebra is unitary. If M is also a VOA module then we say that M is a unitary VOA module for V. V itself is an irreducible unitary VOA module called the adjoint module (the vacuum representation).

Fom VOA modules to representations of conformal nets

Let V be a strongly local VOA and let M be a unitary VOA module for V.

- We assume that M is energy-bounded i.e. that for every $a \in V$ there exist positive integers s_M, j_M and a constant $K_M > 0$ such that

$$\|a_n^M b\| \leq K_M(|n|+1)^{s_M} \|(L_0^M + 1_M)^{j_M} b\| \ \forall n \in \mathbb{Z}, \ \forall b \in M.$$

- Let $\mathcal{H}_M$ be the Hilbert space completion of M and let $f \in C^\infty(S^1)$ with Fourier coefficients $\hat{f}_n$. For every $a \in V$ we define the operator $Y_M^0(a, f)$ on $\mathcal{H}_M$ with domain M by

$$Y_M^0(a, f)b = \sum_{n \in \mathbb{Z}} a_n^M \hat{f}_n b \quad \text{for } b \in M.$$

It is a closable operator and we denote its closure by $Y_M(a, f)$ (smeared vertex operator in the representation).

- Let V be a strongly local VOA and let $\mathcal{A}_V$ be the corresponding conformal net. If π is a (locally normal) representation of V and $I \subset S^1$ is an interval then, the normal representation π_I of the von Neumann algebra $A_V(I)$ on $\mathcal{H}_\pi$ naturally extends to the unbounded operators affiliated with $\mathcal{A}_V(I)$. In particular $\pi_I(Y(a,f))$ is a well defined closed operator on $\mathcal{H}_\pi$ for all $a \in V$ and all $f \in C_c^\infty(I)$.
- Definition [CWX≥2017]. Let M be a unitary energy-bounded VOA module for V. We say that M is strongly integrable if there is a representation π^M of $\mathcal{A}_V$ on $\mathcal{H}_M$ such that $\pi_I^M(Y(a,f)) = Y_M(a,f)$ for all intervals $I \subset S^1$, all $a \in V$ and all $f \in C_c^\infty(I)$.
- Let $\mathrm{Rep}_u(V)$ be the category of unitary VOA modules for V. Then the strongly integrable V-modules define a full subcategory $\mathrm{Rep}_{si}(V)$ of $\mathrm{Rep}_u(V)$ which is closed under subobjects and direct sums. Moreover, let $\mathrm{Rep}(\mathcal{A}_V)$ be the category of (locally normal) representations of $\mathcal{A}_V$.

We have the following results [CWX≥2017]

- The map $M \mapsto \pi^M$ gives rise to a functor $F : \mathrm{Rep}_{si}(V) \to \mathrm{Rep}(\mathcal{A}_V)$.
- M^α and M^β are isomorphic iff π^{M^α} and π^{M^β} are unitarily equivalent.
- The adjoint module V is an irreducible strongly integrable module.
- If $W \subset V$ is a unitary subalgebra and M is a strongly integrable V-module then every VOA W-submodule $\tilde{M} \subset M$ is strongly integrable.
- The map $\tilde{M} \mapsto \pi^{\tilde{M}}$ gives rise to a one-to-one correspondence between the V-submodules $\tilde{M}$ of M and the subrepresentations of π^M. In particular M is irreducible iff π_M is irreducible.

Thanks the above results various examples of strongly integrable modules are given in [CWX≥2017] e.g. those associated to type A affine VOAs. Moreover we can use these results to find a solution to a long standing problem in the representation theory of coset VOAs by using functional analytic methods and in particular the Jones theory of subfactors.

From representations of conformal nets to VOA modules

- Let V be a strongly local VOA and $\mathcal{A}_V$ be the corresponding conformal net. If π is a representation of $\mathcal{A}_V$ it is natural to ask if there is a strongly integrable V-module M such that $\pi = \pi^M$.
- In a work in preparation with M. Weiner [CW$\geq$2018] we use certain local energy bounds to show that if V is a unitary affine VOA then any irreducible representations of $\mathcal{A}_V$ comes from a unitary positive-energy representation of the affine Kac-Moody algebra associated with V.
- Very similar results have been obtained by different methods by Y. Tanimoto [unpublished] and by A. Henriques [arXiv:1706.08471].
- The affine Lie algebra representations gives rise to modules for V and it is natural to expect that these modules are strongly integrable and give back the representations of the net $\mathcal{A}_V$. This is part of an ongoing work by S.C. and M. Weiner.
- The idea of local energy bounds is potentially very general to treat these problems and should have applications beyond affine VOAs, e.g. for certain W-algebras.

- In any case there is still a lot of work to be done.

THANK YOU VERY MUCH!