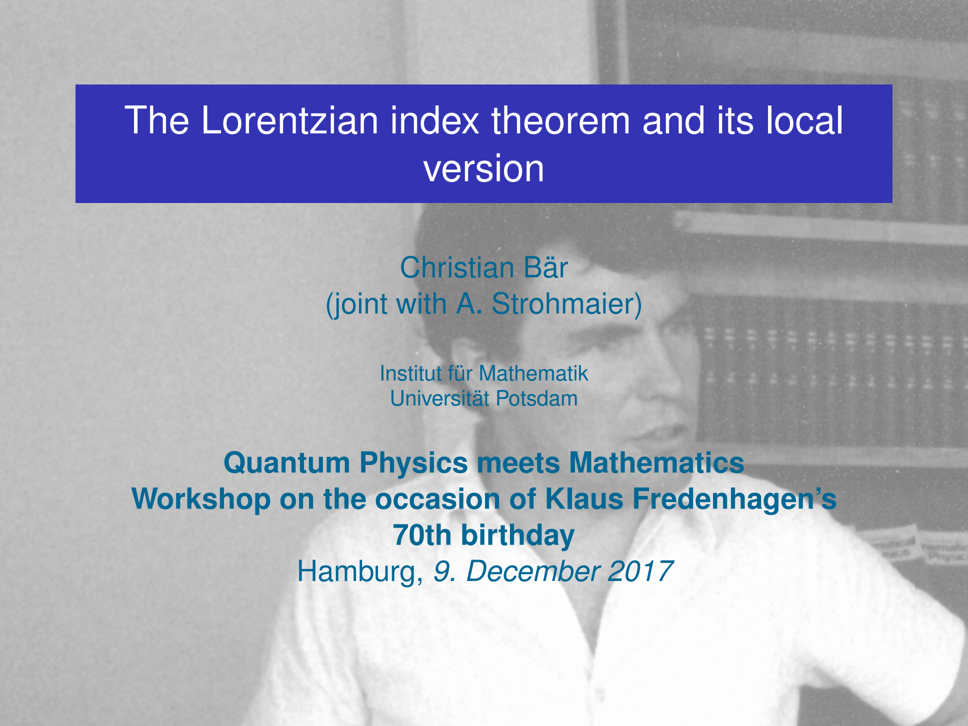

The Lorentzian index theorem and its local version

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Outline

1. Index theory on Riemannian manifolds
   - The (local) Atiyah-Singer index theorem
   - The Atiyah-Patodi-Singer index theorem
2. Index theory on Lorentzian manifolds
   - Dirac operator on Lorentzian manifolds
   - The Lorentzian index theorem
   - The local Lorentzian index theorem

1. Index theory on Riemannian manifolds

Setup

- M Riemannian manifold, compact, without boundary
- spin structure $\rightsquigarrow$ spinor bundle $SM \to M$
- $n = \dim(M)$ even $\rightsquigarrow$ splitting $SM = S_L M \oplus S_R M$
- Hermitian vector bundle $E \to M$ with connection
  $\rightsquigarrow$ twisted spinor bundle $V_{L/R} = S_{L/R} M \otimes E$
- Dirac operator $D : C^\infty(M, V) \to C^\infty(M, V)$

$$D = \begin{pmatrix} 0 & D_R \\ D_L & 0 \end{pmatrix}$$

Properties of D:

- linear differential operator of first order
- elliptic
- essentially self-adjoint
- D_L is Fredholm, i.e. index is defined

$$\operatorname{ind}(D_L) = \dim \ker(D_L) - \dim \operatorname{coker}(D_L)$$

Heat kernel expansion

$D_R D_L$ is of Laplace type $\rightsquigarrow$ heat kernel expansion

$$e^{-tD_R D_L}(x,x) \overset{t\searrow 0}{\sim} (4\pi t)^{-n/2} \sum_{j=0}^{\infty} a_j(x) t^j$$

$\Rightarrow$

$$\mathrm{Tr}(e^{-tD_R D_L}) \overset{t\searrow 0}{\sim} (4\pi t)^{-n/2} \sum_{j=0}^{\infty} \int_M \mathrm{tr}(a_j(x)) t^j$$

Similarly:

$$\mathrm{Tr}(e^{-tD_L D_R}) \overset{t\searrow 0}{\sim} (4\pi t)^{-n/2} \sum_{j=0}^{\infty} \int_M \mathrm{tr}(\tilde{a}_j(x)) t^j$$

Index computation

$$
\begin{aligned}
\mathrm{ind}(D_L) &= \dim\ker(D_L) - \dim\ker(D_R) \\
&= \dim\ker(D_R D_L) - \dim\ker(D_L D_R) \\
&= \sum_k e^{-t\lambda_k(D_R D_L)} - \sum_k e^{-t\lambda_k(D_L D_R)} \\
&= \mathrm{Tr}(e^{-tD_R D_L}) - \mathrm{Tr}(e^{-tD_L D_R}) \\
&\sim (4\pi t)^{-n/2} \sum_{j=0}^{\infty} \int_M [\mathrm{tr}(a_j(x)) - \mathrm{tr}(\tilde{a}_j(x))] t^j
\end{aligned}
$$

$\Rightarrow$

$$
\mathrm{ind}(D_L) = \int_M [\mathrm{tr}(a_{n/2}(x)) - \mathrm{tr}(\tilde{a}_{n/2}(x))],
$$

$$
\int_M [\mathrm{tr}(a_j(x)) - \mathrm{tr}(\tilde{a}_j(x))] = 0 \text{ for } j < n/2
$$

The index theorem

Local index theorem

The following holds pointwise:

$$(4\pi)^{-n/2}[\mathrm{tr}(a_j(x)) - \mathrm{tr}(\tilde{a}_j(x))] = \begin{cases} 0 & \text{for } j < n/2 \\ \hat{\mathrm{A}}(M) \wedge \mathrm{ch}(E)|_x & \text{for } j = n/2 \end{cases}$$

Corollary (Atiyah-Singer 1968)

$$\mathrm{ind}(D_L) = \int_M \hat{\mathrm{A}}(M) \wedge \mathrm{ch}(E)$$

Setup for manifolds with boundary

- M Riemannian manifold, compact, with boundary ∂M
- spin structure $\rightsquigarrow$ spinor bundle $SM \to M$
- $n = \dim(M)$ even $\rightsquigarrow$ splitting $SM = S_L M \oplus S_R M$
- Hermitian vector bundle $E \to M$ with connection
  $\rightsquigarrow$ twisted spinor bundle $V_{L/R} = S_{L/R} M \otimes E$
- Dirac operator $D : C^\infty(M, V) \to C^\infty(M, V)$

$$D = \begin{pmatrix} 0 & D_R \\ D_L & 0 \end{pmatrix}$$

Need boundary conditions:
Let A_0 be the Dirac operator on ∂M.
$P_+ = \chi_{[0,\infty)}(A_0) =$ spectral projector

APS-boundary conditions:

$$P_+(f|_{\partial M}) = 0$$

Atiyah-Patodi-Singer index theorem

Theorem (Atiyah-Patodi-Singer 1975)

Under APS-boundary conditions D_L is Fredholm and

$$\operatorname{ind}(D_L^{\mathrm{APS}}) = \int_M \hat{\mathrm{A}}(M) \wedge \mathrm{ch}(E) + \int_{\partial M} T(\hat{\mathrm{A}}(M) \wedge \mathrm{ch}(E)) - \frac{h(A_0) + \eta(A_0)}{2}$$

Here

- $h(A) = \dim \ker(A)$
- $\eta(A) = \eta_A(0)$ where $\eta_A(s) = \sum_{\substack{\lambda \in \mathrm{spec}(A) \\ \lambda \neq 0}} \mathrm{sign}(\lambda) \cdot |\lambda|^{-s}$

2. Index theory on Lorentzian manifolds

Index theory in Lorentzian signature?

Problem 1: Let D be a differential operator of order k over a closed manifold. Then $D : H^k \to L^2$ is Fredholm $\Leftrightarrow$ D is elliptic.

$\Rightarrow$ no Lorentzian analog to Atiyah-Singer index theorem

Problem 2: Hyperbolic PDEs behave badly on closed manifolds

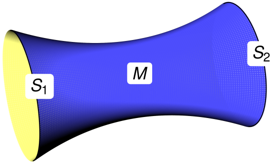
Problem 3: Closed Lorentzian manifolds violate causality conditions

$\Rightarrow$ useless as models in General Relativity

But: There exists a Lorentzian analog to the Atiyah-Patodi-Singer index theorem!

Setup for Lorentzian manifolds

- M globally hyperbolic Lorentzian manifold **with boundary** $\partial M = S_1 \sqcup S_2$
- S_j smooth **compact** spacelike Cauchy hypersurfaces



- spin structure $\rightsquigarrow$ spinor bundle $SM \to M$
- $n = \dim(M)$ even $\rightsquigarrow$ splitting $SM = S_L M \oplus S_R M$
- Hermitian vector bundle $E \to M$ with connection $\rightsquigarrow$ twisted spinor bundles $V_{L/R} = S_{L/R} M \otimes E$
- Dirac operator $D : C^\infty(M, V) \to C^\infty(M, V)$ (hyperbolic!)
- A_j Dirac operator on S_j (elliptic!)

The Lorentzian index theorem

Theorem (B.-Strohmaier 2015)

Under APS-boundary conditions D_L is Fredholm.
The kernel consists of smooth spinor fields and

$$\mathrm{ind}(D_L^{\mathrm{APS}}) = \int_M \widehat{\mathrm{A}}(M) \wedge \mathrm{ch}(E) + \int_{\partial M} T(\widehat{\mathrm{A}}(M) \wedge \mathrm{ch}(E)) - \frac{h(A_1) + h(A_2) + \eta(A_1) - \eta(A_2)}{2}$$

Original proof of the index theorem

Step 1: Show that D_L^{APS} is Fredholm
(microlocal analysis, FIOs)

Step 2: Compute index

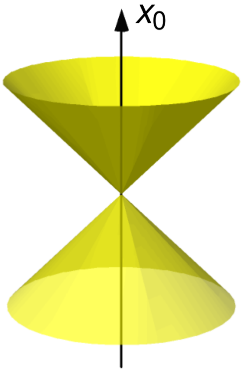
- introduce auxiliary Riemannian metric on M
- use spectral flow to relate the Lorentzian and the Riemannian indices
- apply classical APS theorem

Aim: Replace step 2 by local index theorem

Hadamard type expansion

On Minkowski space $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ define distributions ($k \in \mathbb{N}_0$):

$$H_k = C(k,n)\cdot \lim_{\varepsilon \searrow 0} \begin{cases} (\langle x,x\rangle - i\varepsilon x_0)^{k+1-n/2} & \text{if } k+1 < n/2 \\ \langle x,x\rangle^{k+1-n/2} \log(\langle x,x\rangle - i\varepsilon x_0) & \text{otherwise} \end{cases}$$



Hadamard type expansion

consider as distribution on manifold near x using normal coordinates about x

vary x

$\rightsquigarrow$ distribution $\mathcal{H}_k$ defined on neighborhood $\mathcal{U}$ of diagonal in $M \times M$

$\rightsquigarrow$ formal bisolution of $(D_R D_L)_x u(x,y) = (D_R D_L)^*_y u(x,y) = 0$ on $\mathcal{U}$:

$$u = \sum_{k=0}^{\infty} V_k(x,y)\mathcal{H}_k$$

The Hadamard coefficients V_k are recursively defined and C^∞ on $\mathcal{U}$. Formally:

$$V_k(x,x) = a_k(x)$$

Hadamard solutions

Schwartz kernel theorem

$$\omega \in C^{-\infty}(M \times M) \quad \overset{1:1}{\longleftrightarrow} \quad \hat{\omega} : C_c^{\infty}(M) \to C^{-\infty}(M)$$

$$\hat{\omega}(u)(v) = \omega(u \otimes v)$$

Definition

A bidistribution $\omega \in C^{-\infty}(M \times M)$ is called bisolution if

$$D_R D_L \circ \hat{\omega} = \hat{\omega} \circ D_R D_L = 0.$$

It is called to be of Hadamard form if

$$\omega - \sum_{k=0}^{n/2-1+\ell} V_k \mathcal{H}_k \in C^{\ell}(\mathcal{U}) \qquad \forall \ell \in \mathbb{N}_0.$$

Hadamard regularization

For $\omega \in C^0(\mathcal{U}, V \boxtimes V^*)$ write

$$[\hat{\omega}](x) := \mathrm{tr}(\omega(x,x))$$

For differential operators Q_1, Q_2 of order m_1, m_2 put

$$[Q_1, \omega, Q_2]_{\mathrm{reg}} := [Q_1 \circ \widehat{(\omega - \sum_{k=0}^{N} V_k \mathcal{H}_k)} \circ Q_2]$$

with $N = n/2 - 1 + m_1 + m_2$.

Proposition 1

For Hadamard bisolutions ω we have:

$$[D_R D_L, \omega, 1]_{\mathrm{reg}} = [1, \omega, D_R D_L]_{\mathrm{reg}} = [V_{n/2}],$$
$$[D_L, \omega, D_R]_{\mathrm{reg}} = [\tilde{V}_{n/2}].$$

Product manifolds

Let $M = I \times S$ with metric $-dt^2 + g$. Put

$$f(t) = \frac{1}{2} e^{i\Delta^{1/2}t} \Delta^{1/2}(1 - \chi_{\{0\}}(\Delta)) - it\chi_{\{0\}}(\Delta)$$

where $\Delta = D_S^2$. Now

$$(\hat{\omega}_S u)(t, \cdot) = \int_{\mathbb{R}} f(t-s)u(s, \cdot)ds$$

defines a distinguished Hadamard bisolution ω_S.

Product manifolds

Localize h and η:

$$h_x = [\chi_{\{0\}}(D_S)](x)$$
$$\eta_x = \eta_x(0) \quad \text{where}$$
$$\eta_x(s) = \sum_{\lambda \neq 0} \operatorname{sign}(\lambda)|\lambda|^{-s}|\Phi_\lambda(x)|^2$$

Then

$$\int_M h_x dx = h(D_S) \quad \text{and} \quad \int_M \eta_x dx = \eta(D_S)$$

Proposition 2

In the product case, for the distinguished Hadamard bisolution:

$$[D_L, \omega_S, \not\psi_S]_{\text{reg}}(t, x) = \frac{1}{2}(\eta_x + h_x)$$

Regularized Dirac current

For any Hadamard bisolution ω define regularized Dirac current

$$J^{\omega}_{\mathrm{reg}}(\xi) = [D_L, \omega, \not{\xi}]_{\mathrm{reg}}$$

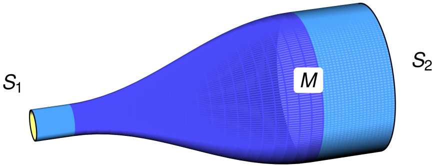
$J^{\omega}_{\mathrm{reg}}$ is a smooth 1-form on M.

In product case, by Proposition 2:

$$J^{\omega_S}_{\mathrm{reg}}(\nu_S) = [D_L, \omega_S, \not{\nu}_S]_{\mathrm{reg}} = \frac{1}{2}(\eta_x + h_x)$$

Crucial computation

Assume M has product metric near boundary $\partial M = S_1 \sqcup S_2$



ω_1: distinguished Hadamard bisolution near S_1, extended to M
propagation of singularities $\Rightarrow$ ω_1 is Hadamard on all of M
similarly for ω_2

Crucial computation

$$
\begin{aligned}
-\operatorname{ind}\left(D_L^{\mathrm{APS}}\right) &= \int_S \left(J^{\omega_1}_{\mathrm{reg}} - J^{\omega_2}_{\mathrm{reg}}\right)(\nu_S)\,dS \qquad \text{(Chiral Anomaly)} \\
&= \int_{S_1} \left(J^{\omega_1}_{\mathrm{reg}} - J^{\omega_2}_{\mathrm{reg}}\right)(\nu_{S_1})\,dS_1 \\
&= \int_{S_1} J^{\omega_1}_{\mathrm{reg}}(\nu_{S_1})\,dS_1 - \int_{S_2} J^{\omega_2}_{\mathrm{reg}}(\nu_{S_2})\,dS_2 + \int_M \delta J^{\omega_2}_{\mathrm{reg}} \\
&= \tfrac{1}{2}(\eta(D_{S_1}) + h(D_{S_1})) - \tfrac{1}{2}(\eta(D_{S_2}) + h(D_{S_2})) + \int_M \delta J^{\omega_2}_{\mathrm{reg}}
\end{aligned}
$$

Thus $\delta J^{\omega_2}_{\mathrm{reg}}$ is the (negative of) the index density.

Local index theorem

Theorem (B.-Strohmaier, 2017)

$$\delta J^{\omega_2}_{\mathrm{reg}} = -\hat{\mathrm{A}}(M) \wedge \mathrm{ch}(E) \qquad \text{pointwise.}$$

Proof: For all $f \in C^\infty_c(\mathring{M})$ we have:

$$\begin{aligned}
\int_M f \cdot \delta J^{\omega_2}_{\mathrm{reg}} &= \int_M J^{\omega_2}_{\mathrm{reg}}(\nabla f) = \int_M [D_L, \omega_2, \not\nabla f]_{\mathrm{reg}} \\
&= \int_M [\not\nabla f \circ D_L, \omega_2, 1]_{\mathrm{reg}} = \int_M [(D_R \circ f - f \circ D_R) \circ D_L, \omega_2, 1]_{\mathrm{reg}} \\
&= \int_M f \cdot \Big([D_L, \omega_2, D_R]_{\mathrm{reg}} - [D_R D_L, \omega_2, 1]_{\mathrm{reg}} \Big) \\
&= \int_M f \cdot \Big([\tilde{V}_{n/2}] - [V_{n/2}] \Big)
\end{aligned}$$

$$\Rightarrow \delta J^{\omega_2}_{\mathrm{reg}} = [\tilde{V}_{n/2} - V_{n/2}] = -\hat{\mathrm{A}}(M) \wedge \mathrm{ch}(E).$$

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Thank you for your attention!