Scattering in QFT without Mass Gaps and Strengthened Reeh-Schlieder Condition

(based on CMP **375**, 2017)

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(joint work with Wojciech Dybalski)

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Technische Universität München

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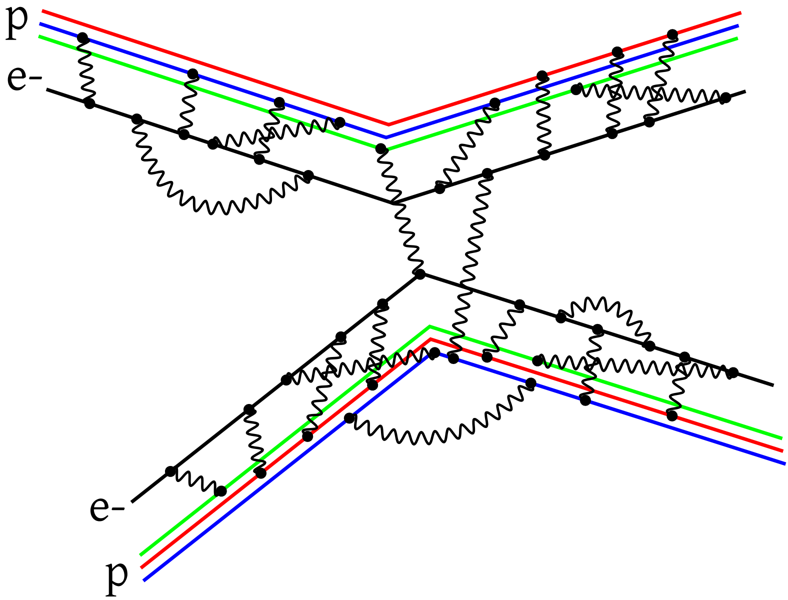
Leipzig, June 23–24, 2017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Interacting Quantum Field Theory, Non-Perturbatively

Exercise 1 Quantum Mechanics:

(a) Find $\mathscr{H}$, Hamiltonian H_0 and Observables for free particles

(b) Born probability interpretation $|\Psi(x)|^2$

(c) Add interaction $H := H_0 + H_{\text{int}}$

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(h) ω defines new Hilbert space $\mathscr{H}$ on which interact. model lives
(change of rep.), and where $H = \lim\limits_{R\to\infty} H^R$ is well-defined.

The Particle Spectrum

Vacuum $\Omega \in \mathscr{H}$ translation invariant, Space-time translations α_x unitarily implemented

$$\mathscr{H} \ni \Psi \longmapsto U(t, \mathbf{x})\Psi$$

SNAG-Theorem $\rightarrow$ strongly commut. self-adjoint generators $(H, \boldsymbol{P})$
$\hat{=}$ energy-momentum op.

Spectral Resolution of $(H, \boldsymbol{P})$ by POVM $E(\Delta)$ for Borel $\Delta \subset \mathbb{R}^4$.

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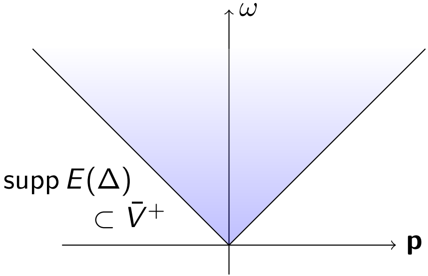

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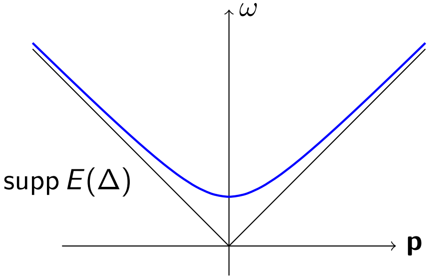

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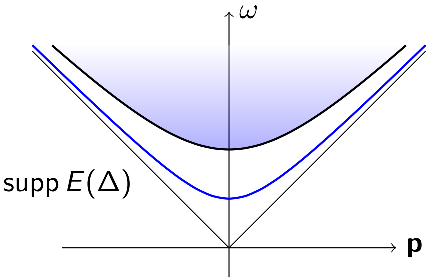

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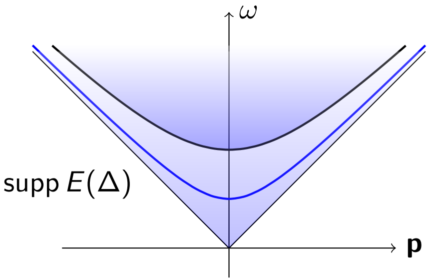

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- Haag '54, Lehmann-Symanzik-Zimmermann '54
  Postulated Asymptotic Condition:
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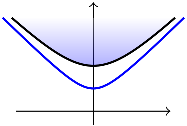
- Ruelle '62, Hepp '65 — proof, isolated mass shell



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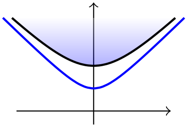

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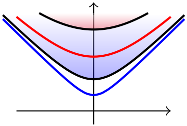

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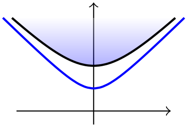

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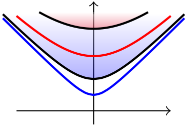

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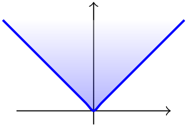

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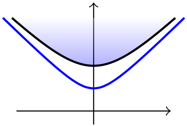

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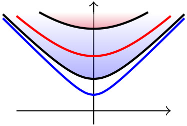

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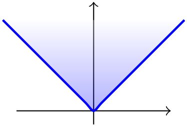
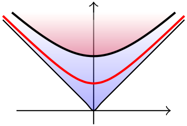


- Dybalski '05 — (SC) + non-isolated vacuum
- Duch, Herdegen '13 — (SC) weakened, $m \geq 0$



Remarks: Other Aspects of the Infrared Problem

Charges, Particles and Infraparticles in AQFT

Our present assumptions restrict us to **neutral**-particle states Ψ_1.

(Electrical) **Charges** are expected to have

- non-sharp particle masses ("Infraparticle")
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- weaker localization properties: [Buchholz'82]
  operators in space-like infinite "strings" $\sim$ *Wilson-lines*

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Current Research and Tentative Approaches

- Scattering of Infraparticles? [Buchholz et al.'91–] [Herdegen'13]
- Space-like asymptotics of $F^{\mu\nu}$ experimentally not accessible, suitable Infravacuum-states conjectured to "stabilize" infraparticles [Kraus, Polley, Reents'77] [Buchholz, Roberts'13]

  $\rightarrow$ Feasible to describe Compton-scattering [Alazzawi, Dybalski'15]
- Perturbation Theory with String-local Quantum Fields [Schroer et al.'04–] [Mund, de Oliveira'16]
- Study infrared problem in more tractable non-relativistic models [Fröhlich'73] [Chen, Fröhlich, Pizzo'07]... [Dybalski, Pizzo'12–]

Non-Locality of the Vacuum: Reeh-Schlieder Property

Local Observables $A \in \mathfrak{A}(\mathcal{O}) \subset B(\mathscr{H}) \sim$ bounded functions of Fields $\phi(x)$ smeared with test functions compactly supported in $\mathcal{O}$.

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Rem.: (HK6$^\flat$) + "Additivity" $\mathfrak{A} \subset \Big(\bigvee_x \mathfrak{A}(\mathcal{O}_0 + x)\Big)'' \Longrightarrow$ (HK6)

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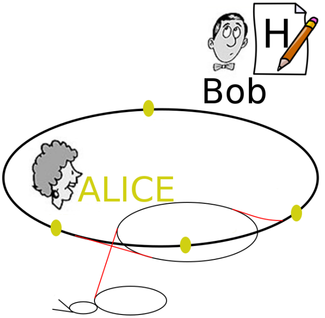

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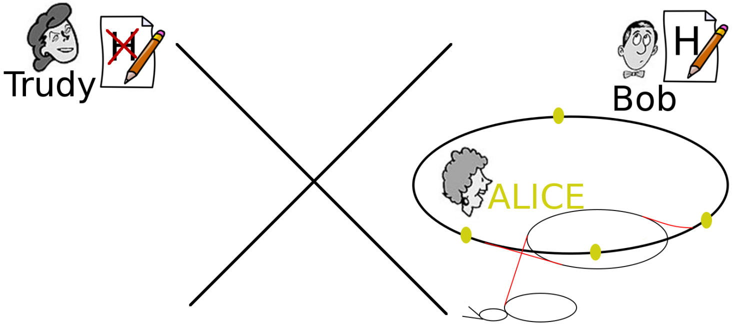

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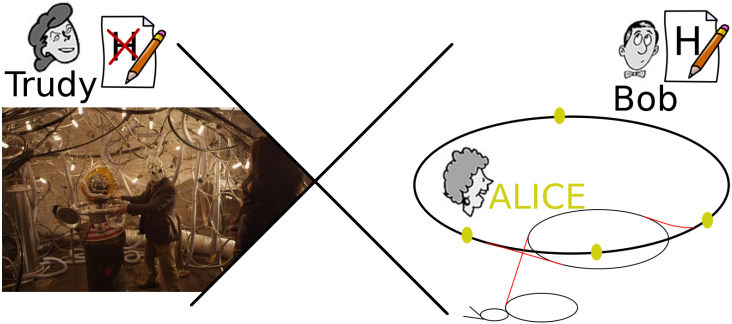

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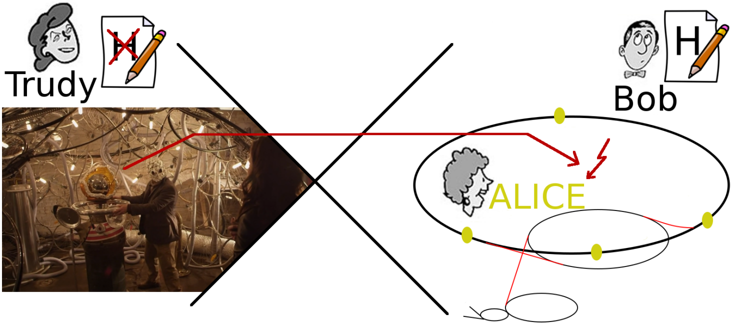

Overview

Algebraic Framework for Local Quantum Theory

Mathematical Objects

Haag-Kastler QFT $(\mathfrak{A}, \alpha, \Omega, \mathscr{H})$ in the vacuum sector.

Described by mathematical entities...

- Hilbert space $\mathscr{H}$ of pure states
- distinguished *vacuum* $\Omega \in \mathscr{H}$
- net of von Neumann algebras $\mathbb{R}^{3+1} \supset \mathcal{O} \mapsto \mathfrak{A}(\mathcal{O}) \subset \mathrm{B}(\mathscr{H})$
- space-time translations of states $(t, \mathbf{x}) \mapsto U(t, \mathbf{x}) = e^{\mathrm{i}tH - \mathrm{i}\mathbf{x}\cdot\boldsymbol{P}}$
- translations of observables $\alpha_x A := A(x) := U(x)\, A\, U(x)^*$

Algebraic Framework for Local Quantum Theory

The Haag-Kastler Axioms

. . . which are subject to

(HK1) $\mathcal{O}_1 \subset \mathcal{O}_2 \Longrightarrow \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$ **(Isotony)**

(HK2) $\mathcal{O}_1 \subset \mathcal{O}_2' \Longrightarrow \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)'$ **(Locality)**

(HK3) $\alpha_x \mathfrak{A}(\mathcal{O}) = \mathfrak{A}(\mathcal{O} + x)$, $\forall x \in \mathbb{R}^4$ **(Covariance)**

(HK4) $E_{(H,\boldsymbol{P})}(\{0\})\mathscr{H} = \mathbb{C}\Omega$ (**Uniqueness of $\boldsymbol{\Omega}$**)

(HK5) $\operatorname{supp} E_{(H,\boldsymbol{P})} \subset \bar{V}^+$ (**Spectrum Condition**)

(HK6) $\overline{\mathfrak{A}(\mathcal{O})\Omega} = \mathscr{H}$ (**Reeh-Schlieder Property**)

Preparing Single-Particle States

Single-particle states $\Psi_1, \Psi_2 \in E_{\{M=m\}}\mathscr{H}$ are non-local objects:

$$\Psi_1 = E_m A\Omega = \chi(\tfrac{M^2-m^2}{\epsilon})A\Omega \sim A(\hat{\chi}_\epsilon)\Omega, \quad (\chi \in \mathscr{S}, \epsilon \searrow 0).$$

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Reeh-Schlieder (HK6) $\Rightarrow \exists (A_{k\beta})_{\beta>0} \subset \mathfrak{A}(\mathcal{O})$: $\|A_{k\beta}\Omega - \Psi_k\| = \beta$.

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Def.: We call a family of local operators $(A_{k\beta})_{\beta>0} \subset \mathfrak{A}(\mathcal{O})$ s.t.

$$\|A_{k\beta}\Omega - \Psi_k\| \leq \beta \text{ and } \|A_{k\beta}\| \leq \beta^{-\gamma}$$

a Reeh-Schlieder family for Ψ_k of degree $\gamma > 0$.

Assumption: **Strengthened Reeh-Schlieder Property** (HK6$^\sharp$)

Reeh-Schlieder families of finite degree generate
a total subset of the single-particle space $\mathscr{H}_1 \subset \mathscr{H}$.

Strengthened Reeh-Schlieder yields Scattering States

Strengthened Reeh-Schlieder Property ($\gamma > 0$)
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Theorem (MD'15) Let Ψ_k be single-particle states admitting Reeh-Schlieder families $A_{k\beta}$ of finite degree. Then for any regular positive-energy Klein-Gordon sol. f_k with disjoint velocity supports

$$\Psi_\tau := \mathcal{B}_{1\tau} \dots \mathcal{B}_{n\tau}\Omega \overset{\tau\to\pm\infty}{\longrightarrow} \Psi^{\pm}.$$

The scalar products of any two such Ψ^+, Ψ'^+ can be computed using the Fock prescription (similarly for incoming states).

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Previous results (Herbst '71, Dybalski '05, Herdegen '13) require spectral condition of Herbst-type, e.g. for some $\epsilon > 0$,

$$\Psi_k = E_{\{M=m\}}A_k\Omega, \quad A_k \in \mathfrak{A}(\mathcal{O}), \quad \left\|E_{\{0<|M-m|<\delta\}}A_k\Omega\right\| \leq \delta^\epsilon.$$

Construction of Scattering States

Reeh-Schlieder and Haag-Ruelle Creation Operators

Reference Dynamics: Klein-Gordon solutions f_k with disjointly and compactly supported wave packets $\tilde{f}_k \in \mathscr{C}_c^\infty(\mathbb{R}^3)$ ("*regular*")

Creation-Operator Approximants: with $\hat{\chi} \in \mathscr{C}_c^\infty(\mathbb{R}^4 \setminus \bar{V}^-)$, set

$$B_{k\beta} := A_{k\beta}(\chi) := \int \mathrm{d}^4x \; \chi(x) \; A_{k\beta}(x),$$

$$\boxed{\mathcal{B}_{k\tau} := \int \mathrm{d}^3x \; f_k(\tau, \mathbf{x}) \; B_{k\beta}(\tau, \mathbf{x}), \quad (\tau \in \mathbb{R}).}$$

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Haag-Ruelle/LSZ: $\mathcal{B}_{k\tau}\Omega \rightharpoonup \Psi'_k(f_k) := \tilde{f}_k(\mathbf{P})\Psi'_k \in \mathscr{H}_1$ for fixed small enough β.

Reeh-Schlieder: $\boxed{\beta = \beta(\tau) := |\tau|^{-\mu}, \; \mu > 0}$ then $\mathcal{B}_{k\tau}\Omega \to \Psi_k(f_k)$.

Candidate Scattering States: Limits $\tau \to \pm\infty$ of $\Psi_\tau := \mathcal{B}_{1\tau}\mathcal{B}_{2\tau}\Omega$.

Mathematical Tools (1) — Discretized Cook's method

$$\|\Psi_{\tau_2} - \Psi_{\tau_1}\| = \left\| \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \, \partial_\tau \Psi_\tau \right\| \leq \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \; \|\partial_\tau \Psi_\tau\| \overset{!}{<} \infty \quad (\tau_2 \to \pm\infty)$$

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For best possible **summability** as $N \to \infty$ we should

- choose $(\tau_k)_{k\in\mathbb{N}}$ as sparse as possible, $\tau_k := (1+\rho)^k \tau_0$, $\rho > 0$
- control equal- and non-equal-time commutators in $(\star\star)$
- control estimation of unbounded leftmost $\mathcal{B}_{j\tau_k}$ in $(\star)$

Tools (2) — Non-Equal-Time Commutator Estimates

$$f_k(t,\mathbf{x}) = \int \mathrm{d}^3k \; \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x} - \mathrm{i}\omega_m(\mathbf{k})t} \; \tilde{f}_k(\mathbf{k}), \quad \tilde{f}_k \in \mathscr{C}_c^\infty(\mathbb{R}^s), \quad \omega_m(\mathbf{k}) := \sqrt{\mathbf{k}^2 + m^2}$$

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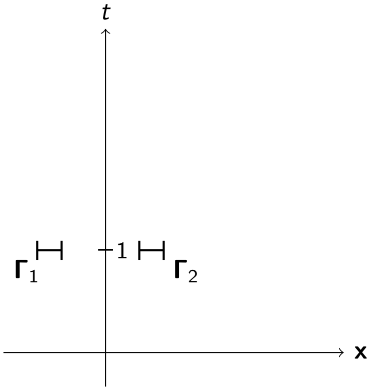

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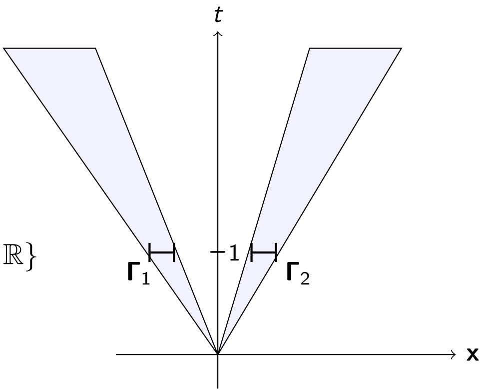

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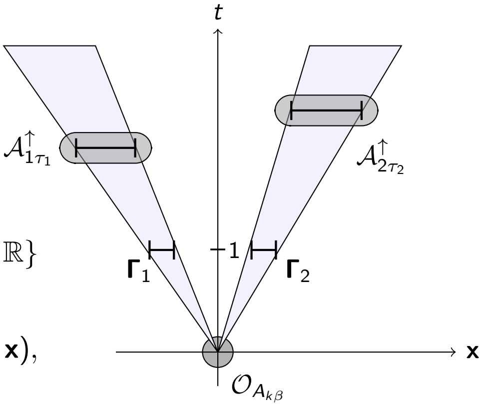

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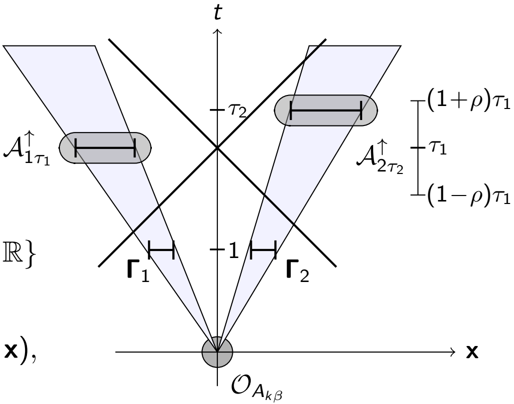

Lemma: Let f_k be regular s.t. $\mathbf{\Gamma}_1 \cap \mathbf{\Gamma}_2 = \emptyset$ and $A_{k\beta}$ have finite degree.

$$\exists\,\rho>0\ \forall\ |\tau_1-\tau_2| \le \rho\,|\tau_1| : \quad \|[\mathcal{B}_{1\tau_1},\mathcal{B}_{2\tau_2}]\| \le \frac{C_N\,\|A_{1\beta(\tau_1)}\|\,\|A_{2\beta(\tau_2)}\|}{1+|\tau_1|^N+|\tau_2|^N}$$

Assembling the Mathematical Arsenal

The reason why Discrete Cook works may be summarized:

Lemma (local difference estimate) Let $A_{k\beta}$ be RS families of finite degree, and f_k regular positive-energy Klein-Gordon solutions with disjoint velocity supports. Then for sufficiently small scaling $\mu > 0$, $\exists\ \rho > 0\ \forall\ |\tau_1 - \tau_2| \leq \rho\,|\tau_1|$,

$$\|\Psi_{\tau_2} - \Psi_{\tau_1}\|^2 \leq C_1 \sum_{k=1}^{n} \|\mathcal{B}_{k\tau_2}\Omega - \mathcal{B}_{k\tau_1}\Omega\|^2 + C_2\,|\tau_1|^{-\delta}$$

Proof based on **non-equal-time** commutator estimates, **energy-bounds** [Buchholz'90], and **Clustering** arguments from [Dybalski'05], [Buchholz'77], and [Araki, Hepp, Ruelle'62].

Is it useful?

Wave Operators and S-Matrix

Let $\mathscr{F}$ denote Fock space over finite RS-degree 1-particle vectors and $\mathscr{F}_{\text{disj}} \subset \mathscr{F}$ the set of product states with disjoint $\mathbf{\Gamma}_k$.

Def. (Møller op.) For $\Psi_{\text{prod}} = \Psi_1(f_1)\Omega \otimes \ldots \otimes \Psi_n(f_n)\Omega \in \mathscr{F}_{\text{disj}}$, $\Psi_k = \lim_{\beta\to 0} \tilde{f}_k(\mathbf{P})A_{k\beta}\Omega$ define

$$W_\pm : \begin{cases} \mathscr{F}_{\text{disj}} \longrightarrow \mathscr{H}, \\ \Psi_{\text{prod}} \longmapsto \lim\limits_{\tau\to\pm\infty} \mathcal{B}_{1\tau} \ldots \mathcal{B}_{n\tau}\Omega. \end{cases}$$

The S-matrix is defined for $\Psi, \Phi \in \mathscr{F}_{\text{disj}}$ by

$$\langle \Psi, S\Phi \rangle := \langle W_+\Psi, W_-\Phi \rangle .$$

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- Proposition. Assume there is a regular local $A \in \mathfrak{A}(\mathcal{O})$ with Herbst-exponent $\epsilon > 0$. Then one can construct $A_\beta \in \mathfrak{A}(\mathcal{O}+B_\epsilon)$ s.t.

$$\|E(\Delta)(A_\beta\Omega - \Psi_1)\| < C_\Delta \beta, \quad \ln \|A_\beta\| < \beta^{-\gamma}$$

  for any compact $\Delta \subset \mathbb{R}^{s+1}$, with suitable C_Δ, and $\gamma \sim 1/\epsilon$.

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Thanks for your attention!