# Gauge theories in curved spacetimes: (Anomalous) Ward identities and the underlying $L_\infty$ algebra

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## Major precursors

- BRST for QED (Dütsch/Fredenhagen, Commun. Math. Phys. 203 (1999) 71, arXiv:hep-th/9807078)
- Master Ward Identity (Dütsch/Boas, Rev. Math. Phys. 14 (2002) 977, arXiv:hep-th/0111101)
- Yang–Mills in curved spacetime (Hollands, Rev. Math. Phys. 20 (2008) 1033, arXiv:0705.3340)
- Batalin–Vilkovisky formalism for closed gauge algebras (Fredenhagen/Rejzner, Commun. Math. Phys. 317 (2013) 697, arXiv:1110.5232)
- Also important: Retarded products (Dütsch/Fredenhagen, Rev. Math. Phys. 16 (2004) 1291, arXiv:hep-th/0403213)

# Perturbative Algebraic Quantum Field Theory

## Perturbative Algebraic Quantum Field Theory 1/5

- Dynamical fields $\{\phi_K\}$, where the index $K$ distinguishes the type of field, and Lorentz, spinor and Lie algebra indices when necessary
- $\epsilon_K \in \{0,1\}$ is the Grassmann parity of $\phi_K$
- Action $S = S_0 + S_{\text{int}}$, with free action $S_0$, interaction $S_{\text{int}}$ at least cubic in the fields
- Free action $S_0 = \frac{1}{2}\int \phi_K(x) P_{KL}(x)\phi_L(x)\,\mathrm{d}x$ with $P_{KL}$ formally self-adjoint: $P^*_{KL} = (-1)^{\epsilon_K\epsilon_L} P_{LK}$
- Unique retarded and advanced Green's functions: $P_{KL}(x) G^{\text{ret/adv}}_{LM}(x,y) = \delta_{KM}\delta(x,y) = P^*_{LK}(y) G^{\text{ret/adv}}_{ML}(x,y)$ with $\operatorname{supp}\int G^{\text{ret/adv}}_{KL}(x,y) f(y)\,\mathrm{d}y \subset J^\pm(\operatorname{supp} f)$ and $G^{\text{adv}}_{KL}(x,y) = (-1)^{\epsilon_K\epsilon_L} G^{\text{ret}}_{LK}(y,x)$
- Pauli–Jordan (commutator) function $\Delta_{KL}(x,y) \equiv G^{\text{ret}}_{KL}(x,y) - G^{\text{adv}}_{KL}(x,y) = G^{\text{ret}}_{KL}(x,y) - (-1)^{\epsilon_K\epsilon_L} G^{\text{ret}}_{LK}(y,x)$

# Perturbative Algebraic Quantum Field Theory 2/5

- $\mathfrak{A}_0$: free $*$-algebra generated by the expressions $\phi_K(f)$, where $f$ is a test function, with the product denoted by $\star_\hbar$, the $*$-relation given by $[\phi_K(f)]^* = \phi_K^\dagger(f^*)$, unit element $\mathbb{1}$, factored by (anti-)commutation relation

$$\begin{aligned}[\phi_K(f), \phi_L(g)]_{\star_\hbar} &\equiv \phi_K(f) \star_\hbar \phi_L(g) - (-1)^{\epsilon_K \epsilon_L} \phi_L(g) \star_\hbar \phi_K(f) \\ &= \mathrm{i}\hbar \int f(x) \Delta_{KL}(x,y) g(y) \, \mathrm{d}x \, \mathrm{d}y \, \mathbb{1} \equiv \mathrm{i}\hbar \Delta_{KL}(f,g) \, \mathbb{1}\end{aligned}$$

- Completion of $\mathfrak{A}_0$ w.r.t. weak topology: free-field algebra $\overline{\mathfrak{A}}_0$
- Practical completion: Consider fixed two-point functions $G^+_{KL}(x,y)$ of Hadamard form, which are bisolutions $P_{KL}(x) G^+_{LM}(x,y) = 0 = P^*_{LK}(y) G^+_{ML}(x,y)$ and satisfy $G^+_{KL}(x,y) - (-1)^{\epsilon_K \epsilon_L} G^+_{LK}(y,x) = \Delta_{KL}(x,y)$ and a certain wave front set condition (microlocal spectrum condition)

## Perturbative Algebraic Quantum Field Theory 3/5

- Normal-ordered products $:\phi_{K_1} \cdots \phi_{K_n}:_G(f_1 \otimes \cdots \otimes f_n) = \int :\phi_{K_1}(x_1) \cdots \phi_{K_n}(x_n):_G f_1(x_1) \cdots f_n(x_n) \, \mathrm{d}x_1 \cdots \mathrm{d}x_n$: defined by $:\phi_K:_G(f) \equiv \phi_K(f)$ and inductively such that

$$
\begin{aligned}
&:\phi_{K_1}(x_1) \cdots \phi_{K_n}(x_n):_G \star_\hbar :\phi_{L_1}(y_1) \cdots \phi_{L_m}(y_m):_G \\
&\quad = :\phi_{K_1}(x_1) \cdots \phi_{K_n}(x_n) \exp\left( \mathrm{i}\hbar \overleftrightarrow{G} \right) \phi_{L_1}(y_1) \cdots \phi_{L_m}(y_m):_G
\end{aligned}
$$

  with $\overleftrightarrow{G} \equiv \int \overleftarrow{\frac{\delta_\mathrm{R}}{\delta\phi_M(u)}} G^+_{MN}(u,v) \overrightarrow{\frac{\delta_\mathrm{L}}{\delta\phi_N(v)}} \, \mathrm{d}u \, \mathrm{d}v$ holds
- Take the limit $f_1(x_1) \otimes \cdots \otimes f_n(x_n) \to f_1(x_1)\delta(x_1, \ldots, x_n)$ (Wick monomials), well-defined thanks to microlocal spectrum condition
- Locally covariant normal products $:\phi_{K_1}(x_1) \cdots \phi_{K_n}(x_n):_H$: use only geometrically defined singular part (Hadamard parametrix $H_{MN}$) instead of two-point function

## Perturbative Algebraic Quantum Field Theory 4/5

- On-shell free-field algebra $\overline{\mathfrak{A}}_0/\mathfrak{I}_0$, where $\mathfrak{I}_0$ is ideal generated by equations of motion $P_{KL}\phi_L = 0$, i.e., elements $\phi_L(P^*_{KL} f)$ and their normal-ordered products
- $\mathcal{F}$: space of local smeared field polynomials (e.g., $\int g(x) F^{\mu\nu}(x) F_{\mu\nu}(x)\, \mathrm{d}x$)
- Time-ordered products: multilinear maps $\mathcal{T}_n \colon \mathcal{F}^{\otimes n} \to \overline{\mathfrak{A}}_0$
- Causal factorisation: $\mathcal{T}_n(F_1 \otimes \cdots \otimes F_n) = \mathcal{T}_\ell(F_1 \otimes \cdots \otimes F_\ell) \star_\hbar \mathcal{T}_{n-\ell}(F_{\ell+1} \otimes \cdots \otimes F_n)$ if $J^+(\operatorname{supp} F_i) \cap J^-(\operatorname{supp} F_j) = \emptyset$ for all $1 \le i \le \ell$, $\ell + 1 \le j \le n$
- Graded symmetry: $\mathcal{T}[\cdots F \otimes G \cdots] = (-1)^{\epsilon_F \epsilon_G} \mathcal{T}[\cdots G \otimes F \cdots]$ for elements $F, G \in \mathcal{F}$ with definite Grassmann parity
- Locality and covariance (cumbersome notation)

## Perturbative Algebraic Quantum Field Theory 5/5

- Non-uniqueness: $\hat{\mathcal{T}}\left[\exp_\otimes\left(\frac{\mathrm{i}}{\hbar}F\right)\right] = \mathcal{T}\left[\exp_\otimes\left(\frac{\mathrm{i}}{\hbar}F + \frac{\mathrm{i}}{\hbar}\mathcal{Z}\left(\mathrm{e}_\otimes^F\right)\right)\right]$ with local and covariant, multilinear maps $\mathcal{Z}_n$ such that
  $\mathcal{Z}_n(F_1 \otimes \cdots \otimes F_n) = 0$ for $\operatorname{supp} F_i \cap \operatorname{supp} F_j = \emptyset$,
  $\mathcal{Z}[\cdots \otimes F \otimes G \otimes \cdots] = (-1)^{\epsilon_F \epsilon_G}\mathcal{Z}[\cdots \otimes G \otimes F \otimes \cdots]$,
  $\mathcal{Z}_n(F^{\otimes n}) = \mathcal{O}(\hbar)$
- Interacting time-ordered products:
  $$\mathcal{T}_L\left[\exp_\otimes\left(\frac{\mathrm{i}}{\hbar}G\right)\right] \equiv \mathcal{T}\left[\exp_\otimes\left(\frac{\mathrm{i}}{\hbar}L\right)\right]^{\star_\hbar(-1)} \star_\hbar \mathcal{T}\left[\exp_\otimes\left(\frac{\mathrm{i}}{\hbar}(L+G)\right)\right]$$
- Smeared interaction: $L = \int g(x)\mathcal{L}\,\mathrm{d}x$ (limit $g \to$ const possible on algebraic level, called “algebraic adiabatic limit”)
- Contrary to appearance, $\mathcal{T}_L$ is formal power series in $\hbar$, special case: interacting field operator corresponding to classical $F$: $\mathcal{T}_L(F)$
- Algebra $\mathfrak{A}$ of interacting quantum fields: formal power series in $\hbar$ with coefficients in $\overline{\mathfrak{A}}_0$

# Gauge theories – the BV-BRST approach

## Gauge theories – the BV-BRST approach 1/3

- Gauge problem: $P_{KL}$ has non-trivial kernel $\Rightarrow$ Batalin–Vilkovisky formalism building on Becchi–Rouet–Stora–Tyutin
- For each symmetry transformation $\delta_\xi$ with parameter $\xi_M$ acting on the fields $\{\phi_K\}$, we introduce a *ghost field* $c_M$, an *antighost field* $\bar{c}_M$, an *auxiliary* field $B_M$ (together $\{\Phi_K\} = \{\phi_K, c_K, \bar{c}_K, B_K\}$)
- Fermionic symmetries (e.g., supersymmetry): $\xi$ is fermionic, global symmetries: no antighost/auxiliary field (non-minimal fields), reducible symmetries: rinse and repeat ("ghosts for ghosts")
- Antifield $\Phi^\ddagger_K$ for each field $\Phi_K$
- New gradings: ghost number $g$, antifield number $a$ such that

$$\epsilon(c_K) = \epsilon(\bar{c}_K) = \epsilon(\xi_K) + 1\,, \quad \epsilon(B_K) = \epsilon(\xi_K)\,, \quad \epsilon(\Phi^\ddagger_K) = \epsilon(\Phi_K) + 1\,,$$
$$g(\phi_K) = 0\,, \quad g(c_K) = 1\,, \quad g(\bar{c}_K) = -1\,, \quad g(B_K) = 0\,,$$
$$g(\Phi^\ddagger_K) = -1 - g(\Phi_K)\,, \quad a(\Phi_K) = 0\,, \quad a(\Phi^\ddagger_K) = 1\,.$$

## Gauge theories – the BV-BRST approach 2/3

- Antibracket: $(F,G) \equiv \int \left( \frac{\delta_{\mathrm{R}} F}{\delta \Phi_K(x)} \frac{\delta_{\mathrm{L}} G}{\delta \Phi^\ddagger_K(x)} - \frac{\delta_{\mathrm{R}} F}{\delta \Phi^\ddagger_K(x)} \frac{\delta_{\mathrm{L}} G}{\delta \Phi_K(x)} \right) \mathrm{d}x$ for $F, G \in \mathcal{F}$ (canonical bracket in field/antifield space)
- Graded symmetry: $(F,G) = (-1)^{\epsilon_F + \epsilon_G + \epsilon_F \epsilon_G} (G,F)$
  Graded Leibniz rule: $(F, GH) = (F,G)H + (-1)^{(1+\epsilon_F)\epsilon_G} G(F,H)$
  Jacobi identity: $(-1)^{(\epsilon_F+1)(\epsilon_H+1)}(F,(G,H)) + \text{cyclic} = 0$
  Grading: $\{g, a, \epsilon\}[(F,G)] = \{g,a,\epsilon\}(F) + \{g,a,\epsilon\}(G) \pm 1$
- BRST differential: $\mathsf{s}F \equiv (S_{\mathrm{tot}}, F)$ with total action $S_{\mathrm{tot}} = S + S_{\mathrm{ext}}$ chosen such that
  1. the *BV master equation* $(S_{\mathrm{tot}}, S_{\mathrm{tot}}) = 0$ is fulfilled,
  2. the original symmetries are recovered by a BRST transformation, with the transformation parameter replaced by the ghost: $\mathsf{s}\phi_M = \sum_k \delta_c \phi_M +$ terms containing antifields,
  3. the non-minimal fields form *trivial pairs*: $\mathsf{s}\bar{c}_M = B_M$, $\mathsf{s}B_M = 0$,
  4. $P_{KL}$ of the antifield-independent free part of $S_{\mathrm{tot}}$ has unique retarded and advanced Green's functions

## Gauge theories – the BV-BRST approach 3/3

- Explicit algorithm to find $S_{\text{ext}}$ as series in antifields, often terminates because of dimensional constraints
- s is odd (fermionic) differential, left derivation (from Leibniz rule), nilpotent $\mathsf{s}^2 = 0$ (from Jacobi identity and BV master equation)
- s augments ghost number by 1, define cohomology classes

$$H^g(\mathsf{s}) \equiv \frac{\mathrm{Ker}(\mathsf{s}\colon \mathcal{F}^g \to \mathcal{F}^{g+1})}{\mathrm{Im}(\mathsf{s}\colon \mathcal{F}^{g-1} \to \mathcal{F}^g)}$$

  $\mathcal{F}^g \subset \mathcal{F}$ subspace of homogeneous elements of ghost number $g$
- $H^0(\mathsf{s})$: classical gauge-invariant observables (representatives independent of trivial pairs, check that also antifield-independent)
- $H^1(\mathsf{s}|\,\mathrm{d})$: obstruction to quantisation, $\mathrm{d}$ is exterior differential
- $H^1(\mathsf{s})$: obstruction for quantum observables

# Anomalous Ward identities

# Anomalous Ward identities 1/9

- Classical theory: symmetry transformation on phase space = Poisson bracket with Noether charge
- Product of classical invariant observables is again invariant, since Poisson bracket obeys Leibniz and classical observables factorise: $(\mathcal{O}_1\mathcal{O}_2)_L = (\mathcal{O}_1)_L(\mathcal{O}_2)_L$, with $(\mathcal{O})_L$ solution of $\dot{\mathcal{O}} = \{\mathcal{O}, L\}$
- In quantum theory: $\mathcal{T}_L(\mathcal{O}_1 \otimes \mathcal{O}_2) \neq \mathcal{T}_L(\mathcal{O}_1) \star_\hbar \mathcal{T}_L(\mathcal{O}_2)$
- Relations between time-ordered products if symmetry is preserved: Ward(–Takahashi–Slavnov–Taylor) identities, but in general extra anomalous terms
- For locally covariant derivation $D$ acting on $\overline{\mathfrak{A}}_0$:

$$D\mathcal{T}\left[\exp_\otimes\left(\frac{\mathrm{i}}{\hbar}F\right)\right] = \frac{\mathrm{i}}{\hbar}\mathcal{T}\left[\left[\mathcal{D}\left(\mathrm{e}_\otimes^F\right) + \mathcal{A}\left(\mathrm{e}_\otimes^F\right)\right] \otimes \exp_\otimes\left(\frac{\mathrm{i}}{\hbar}F\right)\right]$$

# Anomalous Ward identities 2/9

- $\mathcal{D}_n, \mathcal{A}_n$: multilinear maps $\mathcal{F}^{\otimes n} \to \mathcal{F}$
- Graded symmetry:
  $\{\mathcal{D}, \mathcal{A}\}[\cdots \otimes F \otimes G \otimes \cdots] = (-1)^{\epsilon_F \epsilon_G} \{\mathcal{D}, \mathcal{A}\}[\cdots \otimes G \otimes F \otimes \cdots]$
- Support on diagonal: $\{\mathcal{D}, \mathcal{A}\}_n(F_1 \otimes \cdots \otimes F_n) = 0$ if $\operatorname{supp} F_i \cap \operatorname{supp} F_j = \emptyset$ for some $i, j$
- Grading: $g[\{\mathcal{D}, \mathcal{A}\}_n(F_1 \otimes \cdots \otimes F_n)] = d + \sum_{i=1}^n g(F_i)$ if $D \colon \overline{\mathfrak{A}}_0^g \to \overline{\mathfrak{A}}_0^{g+d}$
- Locality and covariance
- Order in $\hbar$: $\mathcal{D}_n(F^{\otimes n}) = \mathcal{O}(\hbar^0)$ ("classical part") and $\mathcal{A}_n(F^{\otimes n}) = \mathcal{O}(\hbar)$ ("anomaly")
- Example: Inner derivation $Da = 1/(\mathrm{i}\hbar)[Q, a]_{\star_\hbar}$ for fixed $Q \in \overline{\mathfrak{A}}_0$ and all $a \in \overline{\mathfrak{A}}_0$ (symmetry obtained by graded commutator with the operator corresponding to the classical Noether charge)

## Anomalous Ward identities 3/9

- Explicit formula for classical part
- If $D$ is an inner derivation, then
  1. Identifying $Q_{\mathrm{cl}} = \lim_{\hbar\to 0} Q$ with an element of $\mathcal{F}$, we have

$$\mathcal{D}_1(F) = \{Q_{\mathrm{cl}}, F\} = \iint \frac{\delta_{\mathrm{R}} Q_{\mathrm{cl}}}{\delta\phi_M(x)} \Delta_{MN}(x,y) \frac{\delta_{\mathrm{L}} F}{\delta\phi_N(y)} \,\mathrm{d}x\,\mathrm{d}y\,.$$

  2. At second order, we have

$$\mathcal{D}_2(F\otimes F) = \iint \frac{\delta_{\mathrm{R}} F}{\delta\phi_K(x)} \left[G^{\mathrm{ret}}_{KL}(x,y) + G^{\mathrm{adv}}_{KL}(x,y)\right] \left\{ \frac{\delta_{\mathrm{L}} Q_{\mathrm{cl}}}{\delta\phi_L(y)}, F \right\} \mathrm{d}x\,\mathrm{d}y\,.$$

  3. $\mathcal{D}_k\left(F^{\otimes k}\right) = 0$ for all $k \geq 3$ if $Q_{\mathrm{cl}}$ is at most of second order in fields, i.e. if $\delta^3 Q_{\mathrm{cl}}/[\delta\phi_K(x)\delta\phi_L(y)\delta\phi_M(z)] = 0$ for all $K, L, M$.

## Anomalous Ward identities 4/9

- If $D$ acts by the antibracket with an element $Q \in \mathcal{F}$ at most of second order in fields (or antifields), that is

$$D\Phi_K(x) = -\frac{\delta_{\mathrm{R}} Q}{\delta \Phi^{\ddagger}_K(x)}\,, \qquad D\Phi^{\ddagger}_K(x) = \frac{\delta_{\mathrm{R}} Q}{\delta \Phi_K(x)}\,,$$

  and extended to general $A \in \overline{\mathfrak{A}}_0$ by linearity and a graded Leibniz rule, then

  1. At first order, we have

  $$\mathcal{D}_1(F) = (Q_{\mathrm{cl}}, F)\,.$$

  2. At second order, we have

  $$\mathcal{D}_2(F \otimes F) = \iint \frac{\delta_{\mathrm{R}} F}{\delta \Phi_K(x)} \left[ G^{\mathrm{ret}}_{KL}(x,y) + G^{\mathrm{adv}}_{KL}(x,y) \right] \left( \frac{\delta_{\mathrm{L}} Q_{\mathrm{cl}}}{\delta \Phi_L(y)}, F \right) \mathrm{d}x \, \mathrm{d}y\,.$$

  3. $\mathcal{D}_k\left(F^{\otimes k}\right) = 0$ for all $k \geq 3$.

## Anomalous Ward identities 5/9

- Application to BRST differential $\mathrm{s}F = (S, F)$: consider free part $\mathrm{s}_0 F = (S_0, F)$ with $S_0$ quadratic in fields and antifields
- Free BRST differential $\hat{\mathrm{s}}_0$ acts on $\overline{\mathfrak{A}}_0$ by $\hat{\mathrm{s}}_0 \Phi_K(x) = -\delta_{\mathrm{R}} S_0 / \delta \Phi^\ddagger_K(x)$, $\hat{\mathrm{s}}_0 \Phi^\ddagger_K(x) = \delta_{\mathrm{R}} S_0 / \delta \Phi_K(x)$, linearity, graded Leibniz rule, and we obtain $\mathcal{D}_2(F \otimes F) = (F, F)$
- Anomalous Ward identity:

$$\hat{\mathrm{s}}_0 \mathcal{T}\left[\exp_\otimes\left(\frac{\mathrm{i}}{\hbar} F\right)\right] = \frac{\mathrm{i}}{\hbar} \mathcal{T}\left[\left(\mathrm{s}_0 F + \frac{1}{2}(F, F) + \mathcal{A}\left(\mathrm{e}_\otimes^F\right)\right) \otimes \exp_\otimes\left(\frac{\mathrm{i}}{\hbar} F\right)\right]$$

- Consistency condition follows from $\hat{\mathrm{s}}_0^2 = 0$:

$$\left(S_0 + F, \mathcal{A}\left[\mathrm{e}_\otimes^F\right]\right) = \frac{1}{2} \mathcal{A}\left[(S_0 + F, S_0 + F) \otimes \mathrm{e}_\otimes^F\right] + \mathcal{A}\left[\mathcal{A}\left[\mathrm{e}_\otimes^F\right] \otimes \mathrm{e}_\otimes^F\right]$$

- If $H^1(\mathrm{s}|d) = \emptyset$, can use freedom in definition of time-ordered products to obtain $\mathcal{A}\left[\mathrm{e}_\otimes^L\right] = 0$ using consistency condition, order by order in $\hbar$

## Anomalous Ward identities 6/9

- For interacting time-ordered products: if $\mathcal{A}\left[\mathrm{e}_\otimes^L\right] = 0$, we have

$$\hat{\mathsf{s}}\mathcal{T}_L\left[\exp_\otimes\left(\frac{\mathrm{i}}{\hbar}F\right)\right] = \frac{\mathrm{i}}{\hbar}\mathcal{T}_L\left[\left(\mathsf{s}F + \frac{1}{2}(F,F) + \mathcal{A}\left(\mathrm{e}_\otimes^{L+F}\right)\right) \otimes \exp_\otimes\left(\frac{\mathrm{i}}{\hbar}F\right)\right]$$

  with $\hat{\mathsf{s}}a \equiv \hat{\mathsf{s}}_0 a + 1/(\mathrm{i}\hbar)[\mathcal{T}_L(\Delta Q^-), a]_{\star_\hbar}$ for $a \in \overline{\mathfrak{A}}_0$ and $\Delta Q^- \in \mathcal{F}$
- Define $n$-ary quantum brackets $[\cdot]_\hbar$:

$$[F_1]_\hbar \equiv \mathsf{s}F_1 + (-1)^{\epsilon_1}\mathcal{A}\left[F_1 \otimes \mathrm{e}_\otimes^L\right],$$

$$[F_1, F_2]_\hbar \equiv (-1)^{\epsilon_1}(F_1, F_2) + (-1)^{\epsilon_1+\epsilon_2}\mathcal{A}\left[F_1 \otimes F_2 \otimes \mathrm{e}_\otimes^L\right],$$

$$[F_1, \ldots, F_k]_\hbar \equiv (-1)^{\epsilon_1+\cdots+\epsilon_k}\mathcal{A}\left[F_1 \otimes \cdots \otimes F_k \otimes \mathrm{e}_\otimes^L\right], \qquad k \geq 3.$$

- Signs ensure *intrinsic oddness*: $[\alpha G, F^k]_\hbar = (-1)^{\epsilon_\alpha}\alpha[G, F^k]_\hbar$, graded symmetry inherited from anomaly terms $\mathcal{A}_n$

# Anomalous Ward identities 7/9

- Interacting Ward identity for $k$ fields:

$$\hat{\mathrm{s}}\mathcal{T}_L\Big[F^{\otimes k}\Big] = \sum_{\ell=1}^{k} \frac{k!}{\ell!(k-\ell)!}\left(\frac{\hbar}{\mathrm{i}}\right)^{\ell-1}\mathcal{T}_L\Big[[F^\ell]_\hbar \otimes F^{\otimes(k-\ell)}\Big]$$

- Nilpotency $\hat{\mathrm{s}}^2 = 0$ implies $\displaystyle\sum_{\ell=1}^{n} \frac{n!}{(n-\ell)!\ell!}[F^{n-\ell}, [F^\ell]_\hbar]_\hbar = 0$
- Intrinsic oddness, graded symmetry of $[\cdot]_\hbar$ and above relation: quantum brackets form an $L_\infty$ algebra over $\mathcal{F}$ (in $b$-picture)
- Level $n = 1$: $[[F]_\hbar]_\hbar = 0$ gives nilpotency of quantum BRST differential $\mathrm{q} \equiv [\cdot]_\hbar = \mathrm{s} + \mathcal{O}(\hbar)$, quantum observables are in $H^0(\mathrm{q})$

## Anomalous Ward identities 8/9

- Level $n = 2$: $2[F, [F]_\hbar]_\hbar + [[F, F]_\hbar]_\hbar = 0$ ensures compatibility of quantum antibracket $(F, G)_\hbar \equiv (-1)^{\epsilon_F}[F, G]_\hbar = (F, G) + \mathcal{O}(\hbar)$ and quantum BRST differential: $\mathrm{q}(F, F)_\hbar = -2(F, \mathrm{q}F)_\hbar$
- Compatibility condition ensures that antibracket is a well-defined map between cohomology classes $(\cdot, \cdot)_\hbar \colon H^g(\mathrm{q}) \otimes H^{g'}(\mathrm{q}) \to H^{g+g'+1}(\mathrm{q})$
- Level $n = 3$: $3[F, F, [F]_\hbar]_\hbar + 3[F, [F, F]_\hbar]_\hbar + [[F^3]_\hbar]_\hbar = 0$ represents the Jacobi identity for the quantum antibracket in cohomology: $(F, (F, F)_\hbar)_\hbar = -[F, F, \mathrm{q}F]_\hbar - \frac{1}{3}\mathrm{q}[F^3]_\hbar = 0 \mod \mathrm{q}$

## Anomalous Ward identities 9/9

- If moreover $H^1(\mathrm{s}) = \emptyset$, we have
  1. To each classical observable corresponds an observable in the quantum theory, that is, each representative of $H^0(\mathrm{s})$ can be extended to a representative of $H^0(\mathrm{q})$.
  2. There exist maps $\mathcal{C}_n \colon \mathcal{F}^{0\otimes n} \to \mathcal{F}^0$, $n \geq 1$ (the contact terms), such that the interacting time-ordered product

$$\mathcal{T}_L\left[\exp_\otimes\left[\frac{\mathrm{i}}{\hbar}F - \frac{\mathrm{i}}{\hbar}\mathcal{C}\left(\mathrm{e}_\otimes^F\right)\right]\right]$$

is independent of the choice of representative $F \in H^0(\mathrm{q})$, up to $\hat{\mathrm{s}}$-exact terms. They satisfy the identities

$$\left[\exp\left(F - \mathcal{C}\left(\mathrm{e}_\otimes^F\right)\right)\right]_\hbar = 0\,,$$

$\mathcal{C}_1(F) = 0$, and

$$\mathcal{C}\left(\mathrm{e}_\otimes^F \otimes \mathrm{q}\, G\right) = \left[1 - \exp\left(F - \mathcal{C}\left(\mathrm{e}_\otimes^F\right)\right), G\right]_\hbar$$

for $G \in \mathcal{F}^{-1}$.

## Conclusions

- You are overwhelmed and absolutely fascinated, but it's time to go home.
- There exists an anomalous Ward identity for each derivation on the algebra of perturbatively interacting quantum fields, encoding violations of the classically expected result.
- For the BRST differential in quantum gauge theories, the anomalous terms in this Ward identity form an $L_\infty$ algebra.
- The relations of this $L_\infty$ algebra ensure that time-ordered products are independent of the choice of representative for an observable.
- I want to use this to show that nice observables exist in quantum gravity, i.e., that they are renormalisable – lack of time prevented me from actually doing it so far.

## Thank you for your attention

Questions?

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