Bisognano-Wichmann property in asymptotically complete massless QFT

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Motivation

The transformations:

1. Parity: $P(t, \vec{x}) = (t, -\vec{x})$,
2. Time reversal: $T(t, \vec{x}) = (-t, \vec{x})$,
3. Charge conjugation: $C\{\text{particle}\} \mapsto \{\text{antiparticle}\}$,

are not necessarily symmetries of physical theories.

1. However, there is strong evidence that CPT is a symmetry.
2. In mathematical QFT various CPT theorems are available. [Lüders 54, Pauli 55, Jost 57,... Guido-Longo 95].
3. Bisognano-Wichmann (BW) property is an assumption in modern CPT theorems.

Outline

1. Relativistic Quantum Mechanics
   - Poincaré group and its massless irreps U_s
   - Modularity condition (MC)
   - Proof of MC for $U_s \oplus U_{-s}$
2. Algebraic QFT
   - Bisognano-Wichmann (BW) property
   - MC$\Rightarrow$ BW at the single-particle level
   - Collision theory and full BW
3. Conclusion: BW$\Rightarrow$ CPT

Lorentz group

Minkowski spacetime: $(\mathbb{R}^4, \eta)$ with $\eta := \mathrm{diag}(1, -1, -1, -1)$.

1. Lorentz group: $\mathcal{L} := \mathrm{O}(1,3) := \{\, \Lambda \in \mathrm{GL}(4, \mathbb{R}) \,|\, \Lambda \eta \Lambda^{\mathrm{T}} = \eta \,\}$
2. Proper ortochronous Lorentz group: $\mathcal{L}_+^\uparrow$ - connected component of unity in $\mathcal{L}$.

$$\mathcal{L} = \mathcal{L}_+^\uparrow \cup T\mathcal{L}_+^\uparrow \cup P\mathcal{L}_+^\uparrow \cup TP\mathcal{L}_+^\uparrow,$$

   where $T(x^0, \vec{x}) = (-x^0, \vec{x})$ and $P(x^0, \vec{x}) = (x^0, -\vec{x})$.
3. Covering group: $\widetilde{\mathcal{L}}_+^\uparrow = \mathrm{SL}(2, \mathbb{C}) = \{\lambda \in \mathrm{GL}(2, \mathbb{C}) \,|\, \det \lambda = 1\}$

Poincaré group

1. Poincaré group: $\mathcal{P} := \mathbb{R}^4 \rtimes \mathcal{L}$.
2. Proper ortochronous Poincaré group: $\mathcal{P}_+^\uparrow := \mathbb{R}^4 \rtimes \mathcal{L}_+^\uparrow$.
3. Covering group: $\widetilde{\mathcal{P}}_+^\uparrow = \mathbb{R}^4 \rtimes \widetilde{\mathcal{L}}_+^\uparrow = \mathbb{R}^4 \rtimes \mathrm{SL}(2, \mathbb{C})$

Symmetries of a quantum theory

1. $\mathcal{H}$ - complex Hilbert space of physical states.
2. For $\Psi \in \mathcal{H}$, $\|\Psi\| = 1$ define the ray $\hat{\Psi} := \{ \mathrm{e}^{\mathrm{i}\phi}\Psi \,|\, \phi \in \mathbb{R} \}$.
3. $\hat{\mathcal{H}}$ - set of rays with the ray product $[\hat{\Phi}|\hat{\Psi}] := |\langle \Phi, \Psi \rangle|^2$.

Definition

A symmetry of a quantum system is an invertible map $\hat{S} : \hat{\mathcal{H}} \to \hat{\mathcal{H}}$ s.t. $[\hat{S}\hat{\Phi}|\hat{S}\hat{\Psi}] = [\hat{\Phi}|\hat{\Psi}]$.

Symmetries of a quantum theory

Theorem (Wigner 31)

For any symmetry transformation $\hat{S} : \hat{\mathcal{H}} \to \hat{\mathcal{H}}$ we can find a unitary or anti-unitary operator $S : \mathcal{H} \to \mathcal{H}$ s.t. $\hat{S}\hat{\Psi} = \widehat{S\Psi}$. S is unique up to phase.

Application:

1. $\mathcal{P}_+^\uparrow$ is a symmetry of our theory i.e., $\mathcal{P}_+^\uparrow \ni (a, \Lambda) \mapsto \hat{S}(a, \Lambda)$.

2. Thus we obtain a projective unitary representation S of $\mathcal{P}_+^\uparrow$

$$S(a_1, \Lambda_1)S(a_2, \Lambda_2) = \mathrm{e}^{i\varphi_{1,2}} S((a_1, \Lambda_1)(a_2, \Lambda_2)).$$

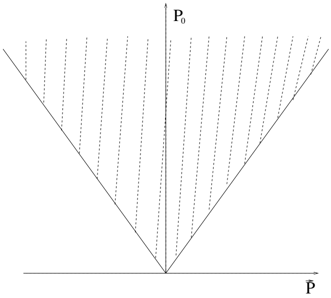
3. Fact: A projective unitary representation of $\mathcal{P}_+^\uparrow$ corresponds to an ordinary unitary representation of the covering group

$$\widetilde{\mathcal{P}}_+^\uparrow \ni (a, \lambda) \mapsto U(a, \lambda) \in B(\mathcal{H}).$$

Positivity of energy

Consider a unitary representation $\widetilde{\mathcal{P}}_+^\uparrow \ni (a, \lambda) \mapsto U(a, \lambda) \in B(\mathcal{H})$.

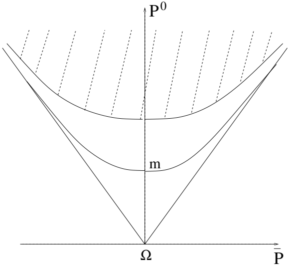
1. $P^\mu := \mathrm{i}^{-1}\partial_{a_\mu} U(a, I)|_{a=0}$ - energy momentum operators.
2. If $\mathrm{Sp}\, P \subset \overline{V}_+$ then we say that U has positive energy.



Distinguished states

1. Def: $\Omega \in \mathcal{H}$ is the vacuum state if $U(a,\lambda)\Omega = \Omega$ for all $(a,\lambda) \in \widetilde{\mathcal{P}}_+^\uparrow$.

2. Def: $\mathcal{H}^{(1)} \subset \mathcal{H}$ is the subspace of single-particle states of mass m and spin s if $U \restriction \mathcal{H}^{(1)}$ is a finite direct sum of irreducible representations $[m,s]$. E.g. for photons: $[0,1] \oplus [0,-1]$.



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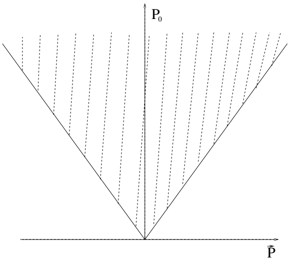

Structure of $[m = 0, s]$ representations of $\widetilde{\mathcal{P}}_+^\uparrow = \mathbb{R}^4 \rtimes \widetilde{\mathcal{L}}_+^\uparrow$

1. Fix a vector at the boundary of the lightcone, e.g. $q = (1, 1, 0, 0)$.
2. Fact: the stabilizer of q in $\widetilde{\mathcal{L}}_+^\uparrow$ is $\mathrm{Stab}_q = \widetilde{\mathrm{E}}(2)$.
3. Def. $\mathrm{Stab}_q \ni (y, \phi) \mapsto V_s(y, \phi) = e^{i\phi s}$, $s \in \mathbb{Z}/2$, is a representation of finite spin s.
4. Def. The $[m = 0, s]$ representation of $\widetilde{\mathcal{P}}_+^\uparrow$ on $L^2(\partial V_+)$:

$$(U_s(a, \lambda)\psi)(p) = e^{ipa} V_s(b_p \lambda b_{\Lambda(\lambda)^{-1}p})\psi(\Lambda(\lambda)^{-1}p),$$

where $\Lambda : \widetilde{\mathcal{L}}_+^\uparrow \to \mathcal{L}_+^\uparrow$ is the covering map and $\Lambda(b_p)q = p$.

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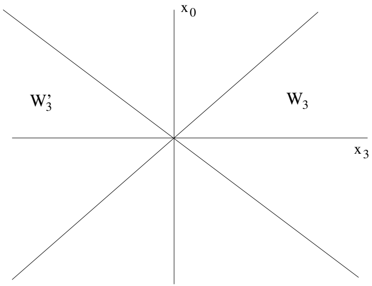
4. Def. The $[m=0,s]$ representation of $\widetilde{\mathcal{P}}_+^\uparrow$ on $L^2(\partial V_+)$:

$$U_s = \mathrm{Ind}_{\mathbb{R}^4\rtimes \mathrm{Stab}_q}^{\widetilde{\mathcal{P}}_+^\uparrow}(q\cdot V_s)$$

Modularity condition (MC)

First, we introduce a wedge $W_3 = \{ x \in \mathbb{R}^4 \, : \, |x_0| < x_3 \}$ in Minkowski spacetime and the opposite wedge W_3'



Modularity condition (MC)

1. Def: G_3^0 is the subgroup of $\lambda \in \widetilde{\mathcal{L}}_+^\uparrow$ s.t. $\Lambda(\lambda)W_3 = W_3$.
2. Def: $G_3 = \langle G_3^0, \mathbb{R}^4 \rangle$.
3. Def: $r_1(\pi) \in \widetilde{\mathcal{L}}_+^\uparrow$ is the rotation around the 1st axis. In particular, $\Lambda(r_1(\pi))W_3 = W_3'$.
4. Def: $\hat{G}_3 = \langle G_3, r_1(\pi) \rangle$.
5. Def: A $\hat{G}_3$-representation $\hat{U}$ satisfies the modularity condition (MC) if $\hat{U}(r_1(\pi)) \in \hat{U}(G_3)''$. [Morinelli 18]
6. As we will discuss later, MC $\Rightarrow$ BW $\Rightarrow$ CPT

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2. Fact [Morinelli 18]: If $\hat{U}$ satisfies MC then $\hat{U} \otimes 1_{\mathcal{K}}$ satisfies MC.
3. Fact [Morinelli 18]: $U_s|_{\hat{G}_3}$, $s \in \mathbb{Z}/2$, satisfy MC.

Theorem (Morinelli-W.D. 19)

Representations $(U_s \oplus U_{-s})|_{\hat{G}_3}$, $s \in \mathbb{Z}$, *satisfy MC.*

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Idea of proof:

1. We show $U_s|_{\hat{G}_3} \simeq U_{-s}|_{\hat{G}_3}$.
2. Then $(U_s \oplus U_{-s})|_{\hat{G}_3} \simeq U_s|_{\hat{G}_3} \otimes 1_{\mathbb{C}^2}$, hence it satisfies MC.

Proof of $U_s|_{\hat{G}_3} \simeq U_{-s}|_{\hat{G}_3}$.

Recall that U_s is an induced representation:

$$U_s = \mathrm{Ind}_{\mathbb{R}^4 \rtimes Stab_q}^{\tilde{\mathcal{P}}_+^\uparrow} (q \cdot V_s), \text{ where } \mathrm{Stab}_q = \widetilde{E}(2), \ V_s(y, \phi) = e^{i\phi s}.$$

We apply the Mackey subgroup theorem:

1. Let $H_1, H_2 \subset G$ be (suitable) closed subgroups.
2. Let ρ be a representation of H_1.
3. Then $(\mathrm{Ind}_{H_1}^G \rho)|_{H_2} \simeq \int_{H_1 \backslash G / H_2}^{\oplus} \mathrm{Ind}_{H_g}^{H_2} (\rho \circ \mathrm{Ad}\, g) \, d\nu([g])$,

   where $H_g := H_2 \cap (g^{-1} H_1 g)$.

Application: $U_s|_{\hat{G}_3} \simeq \int_{\mathbb{R}^+}^{\oplus} \mathrm{Ind}_{\mathbb{R}^4 \rtimes \langle r_1(\pi) \rangle}^{\hat{G}_3} (rq \cdot V_s) dr \simeq U_{-s}|_{\hat{G}_3}$.

Relativistic Quantum Mechanics

Definition

A relativistic quantum mechanical theory is given by:

1. $\mathcal{H}$ - Hilbert space.
2. $\widetilde{\mathcal{P}}_+^\uparrow \ni (a, \lambda) \mapsto U(a, \lambda) \in B(\mathcal{H})$ - a positive energy unitary rep.
3. $B(\mathcal{H})$ - possible observables.

$\mathcal{H}$ may contain a vacuum state Ω and/or subspaces of single-particle states $\mathcal{H}^{(1)}$.

Relativistic (algebraic) QFT

Definition

A relativistic QFT is a relativistic QM $(U, \mathcal{H})$ with a net

$$\mathbb{R}^4 \supset \mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) \subset B(\mathcal{H})$$

of algebras of observables $\mathcal{A}(\mathcal{O})$ localized in open bounded regions of spacetime $\mathcal{O}$, which satisfies:

1. (Isotony) $\quad \mathcal{O}_1 \subset \mathcal{O}_2 \quad \Rightarrow \quad \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$,
2. (Locality) $\quad \mathcal{O}_1 \sim \mathcal{O}_2 \quad \Rightarrow \quad [\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = \{0\}$,
3. (Covariance) $U(a, \lambda)\mathcal{A}(\mathcal{O})U(a, \lambda)^* = \mathcal{A}(\Lambda(\lambda)\mathcal{O} + a)$.

Furthermore, there is a vacuum vector Ω, cyclic for $\mathcal{A} := \overline{\bigcup_{\mathcal{O}\subset\mathbb{R}^4} \mathcal{A}(\mathcal{O})}$.

Bisognano-Wichmann property

1. $W_3 = \{ x \in \mathbb{R}^4 : |x_0| < x_3 \}$ a wedge.
2. $\mathcal{A}(W_3)$ is the von Neumann algebra of this wedge.
3. Tomita-Takesaki theory: $SA\Omega := A^*\Omega$ for $A \in \mathcal{A}(W_3)$.
4. Polar decomposition: $S = J\Delta^{1/2}$.
5. Modular evolution $\mathbb{R} \ni t \mapsto \Delta^{it} = e^{i \log(\Delta) t}$.
6. Def: An algebraic QFT $(\mathcal{A}, U, \Omega)$ has a Bisognano-Wichmann (BW) property if

$$U(\lambda_t) = \Delta^{-it},$$

where $\lambda_t \in \widetilde{\mathcal{L}}_+^{\uparrow}$ is a family of boosts in the direction of the wedge.

Main result

Theorem (Morinelli-W.D. 19)

For algebraic QFT which

1. *describe massless Wigner particles with spins* $(s, -s)$, $s \in \mathbb{Z}$,
2. *are asymptotically complete,*

the Bisognano-Wichmann property holds.

Collision theory and asymptotic completeness

1. Def. An algebraic QFT describes Wigner particles of mass $m = 0$ and spins $(s, -s)$ if there is a subspace $\mathcal{H}^{(1)} \subset \mathcal{H}$ s.t. $U|_{\mathcal{H}^{(1)}} = U_s \oplus U_{-s}$.

2. Def. For $A \in \mathcal{A}(\mathcal{O})$, outgoing asymptotic fields are given by:

$$A_t := -2\,t \int d\omega(\boldsymbol{n})\partial_0 A(t, t\boldsymbol{n}), \quad \overline{A}_t := \frac{1}{\ln t}\int_t^{t+\ln t} dt'\, A_{t'}$$

$$A^{\mathrm{out}} := \lim_{t\to\infty} \overline{A}_t. \qquad \text{Fact: } A^{\mathrm{out}}\Omega \in \mathcal{H}^{(1)}.$$

3. Def. $\mathcal{A}^{\mathrm{out}}(\mathcal{O}) := \{\, e^{iA^{\mathrm{out}}} \;:\; A \in \mathcal{A}_0(\mathcal{O}),\; A^* = A \,\}''$.

4. Fact: $(\mathcal{A}^{\mathrm{out}}, U, \Omega)$ satisfies all the standard properties, with a possible exception of cyclicity of the vacuum. [Buchholz 77]

5. Def. If cyclicity of the vacuum holds, we say that $(\mathcal{A}, U, \Omega)$ is asymptotically complete.

Asymptotic creation/annihilation operators

1. Def: Let $\eta \in S(\mathbb{R}^4)$ be s.t. $\mathrm{supp}\,\widetilde{\eta} \cap \overline{V}_+ = \emptyset$. Then the asymptotic annihilation operators are given by

$$A^{\mathrm{out}-} := \int d^4x\, A^{\mathrm{out}}(x)\eta(x),$$

2. The asymptotic creation operators are given by $A^{\mathrm{out}+} = (A^{\mathrm{out}-})^*$.

3. Scattering states:

$$\Psi^{\mathrm{out}} := A_1^{\mathrm{out}+} \dots A_n^{\mathrm{out}+}\Omega.$$

4. Asymptotic completeness: Scattering states span $\mathcal{H}$.

Main result

Theorem (Morinelli-W.D. 19)

For algebraic QFT which

1. *describe massless Wigner particles with spins $(s, -s)$, $s \in \mathbb{Z}$,*
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the Bisognano-Wichmann property holds.

Proof (idea): Set $Z_t = \Delta^{it} U(\lambda_t)$.

1. By MC, we know that $Z_t A^{\mathrm{out}+}\Omega = A^{\mathrm{out}+}\Omega$.
2. For 2-particle states we write

$$
\begin{aligned}
Z_t A_1^{\mathrm{out}+} A_2^{\mathrm{out}+}\Omega &= (Z_t A_1^{\mathrm{out}+} Z_t^*) A_2^{\mathrm{out}+}\Omega \\
&= [(Z_t A_1^{\mathrm{out}+} Z_t^*), A_2^{\mathrm{out}+}]\Omega + A_2^{\mathrm{out}+} A_1^{\mathrm{out}+}\Omega
\end{aligned}
$$

3. The commutator is zero by explicit computation. $\square$

CPT theorem

Theorem (Lüders 54, Pauli 55, Jost 57,...Guido-Longo 95)

In algebraic QFT satisfying the Bisognano-Wichmann property there exists an anti-unitary operator θ on $\mathcal{H}$ which has the expected properties of the CPT operator, i.e.,

1. $\theta \mathcal{A}(\mathcal{O}) \theta^* = \mathcal{A}(-\mathcal{O})$,
2. $\theta U(a, \lambda) \theta^* = U(-a, \lambda)$,
3. $\theta \rho(\,\cdot\,) \theta^* = \bar{\rho}(\cdot)$ *for DHR morphisms* ρ.

Recall:

1. BW property: $\Delta^{-it} = U(\lambda_t)$,
2. $S = J\Delta^{1/2}$ is defined by $SA\Omega = A^*\Omega$, $A \in \mathcal{A}(W_3)$,

One checks that $\theta := JU(r_3(\pi))^{-1}$ has the required properties.

Conclusions

1. The Bisognano-Wichmann property enters as an assumption in modern CPT theorems.
2. We proved the Bisognano-Wichmann property for asymptotically complete theories of massless particles with spins $(s, -s)$, $s \in \mathbb{Z}$. (The massive case settled by [Mund 01]).
3. Future direction: generalization to fermions, i.e. $s \in \mathbb{Z}/2$.

V. Morinelli, W.D. *The Bisognano-Wichmann property for asymptotically complete massless QFT*. arXiv:1909.12809.