Infinite Index Extensions of Local Nets and Defects

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Plan of the talk

- Preliminaries
- How to "describe" discrete extensions of QFTs (local nets)?
- Braided Products
- Transparent boundary conditions for CFTs

see [arXiv:1703.03605]

Algebraic Quantum Field Theory (AQFT) [Haag, Kastler 64]:

$$\{\mathcal{O} \in \text{spacetime regions} \longmapsto \mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})\}$$

$\mathcal{O}$ = e.g. double cones in $\mathbb{R}^{3+1}$, or intervals in $\mathbb{R}$
$\mathcal{H}$ = Hilbert space of the quantum theory,
$\mathcal{A}(\mathcal{O})$ = von Neumann algebra generated by observables localized in $\mathcal{O}$ (local algebras)

$\rightsquigarrow$ net of local observables or local net, denoted by $\{\mathcal{A}\}$, if

- $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}(\tilde{\mathcal{O}})$ if $\mathcal{O} \subset \tilde{\mathcal{O}}$ (isotony)
- $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}(\tilde{\mathcal{O}})'$ if $\mathcal{O}$ and $\tilde{\mathcal{O}}$ are space-like separated (locality)
- $g \mapsto U(g)$ unitary representation of a spacetime symmetry group $\mathcal{P}$, e.g. Poincaré or Möbius group, such that $U(g)\mathcal{A}(\mathcal{O})U(g)^* = \mathcal{A}(g\mathcal{O})$ (covariance)
- $\Omega \in \mathcal{H}$ ground state of the energy, $U(g)\Omega = \Omega$ (vacuum vector) and $\overline{\bigvee_{\mathcal{O}} \mathcal{A}(\mathcal{O})\Omega} = \mathcal{H}$
- Positive energy condition

"Index" for subfactors [Jones 83]:

- Subfactor: $\mathcal{N} \subset \mathcal{M}$
  where $\mathcal{N}$, $\mathcal{M}$ von Neumann algebras in $\mathcal{B}(\mathcal{H})$ = bounded linear op's on $\mathcal{H}$, and $\mathcal{N}$, $\mathcal{M}$ have trivial center (factors).

- Index: $\mathrm{Ind}(\mathcal{N} \subset \mathcal{M}) \geq 1$ "relative dimension of $\mathcal{M}$ wrt $\mathcal{N}$"

$$\mathcal{N} \subset \mathcal{M} \longmapsto \mathrm{Ind}(\mathcal{N} \subset \mathcal{M})$$

- invariant for subfactors ⇒ classification results
- quantization for small admissible values (Jones' rigidity theorem)
  $\mathrm{Ind}(\mathcal{N} \subset \mathcal{M}) \in \{4\cos^2(\frac{\pi}{n}), n = 3, 4, \ldots\} \cup [4, \infty]$

Extensions of QFTs are "nets of subfactors" [Longo, Rehren 95] :

$$\{\mathcal{A} \subset \mathcal{B}\}$$

i.e.

$$\mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{O})$$

is a subfactor (or inclusion of von Neumann algebras) for *every* $\mathcal{O}$.
(Hilbert space $\mathcal{H}$ is fixed).

Additionally require:

- $\exists\ E$ standard (normal, faithful) conditional expectation from $\{\mathcal{B}\}$ to $\{\mathcal{A}\}$ i.e. $\exists$ a family $E_{\mathcal{O}} \in E(\mathcal{B}(\mathcal{O}), \mathcal{A}(\mathcal{O}))$ with $E_{\mathcal{O}_2 \restriction \mathcal{B}(\mathcal{O}_1)} = E_{\mathcal{O}_1}$ for $\mathcal{O}_1 \subset \mathcal{O}_2$
- Vacuum state $\omega(\cdot) = (\Omega, \cdot\Omega)$ is preserved by E, i.e. $\omega = \omega \circ E$.

How to encode inclusion $\mathcal{N} \overset{E}{\subset} \mathcal{M}$ with data of $\mathcal{N}$?

When $\mathsf{Ind}(\mathcal{N} \subset \mathcal{M}) < \infty$ ($\mathcal{N}, \mathcal{M}$ infinite factors) can use Q-systems in $\mathsf{End}(\mathcal{N})$ [Longo 94].

$\rightsquigarrow$ Purely categorical notion.

Same for inclusions of nets $\{\mathcal{A} \subset \mathcal{B}\}$ with finite index. Use Q-systems in $\mathrm{DHR}\{\mathcal{A}\}$ (Representation category of $\{\mathcal{A}\}$) [Longo, Rehren 95].

Definition (Longo 94)

A Q-system in a simple C^* tensor category $\mathcal{C}$ is a triple (θ, w, x)

with $\theta \in Obj(\mathcal{C})$, $\quad w \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{id}, \theta)$, $\quad x \in \mathrm{Hom}_{\mathcal{C}}(\theta, \theta^2)$

satisfying the following properties:

- Unit property: $(w^* \times 1_\theta) \circ x = (1_\theta \times w^*) \circ x = 1_\theta$
- Associativity: $(x \times 1_\theta) \circ x = (1_\theta \times x) \circ x$
- Frobenius property: $(1_\theta \times x^*) \circ (x \times 1_\theta) = x \circ x^* = (x^* \times 1_\theta) \circ (1_\theta \times x)$
- Standardness: $w^* \circ w = \sqrt{\dim(\theta)} 1_{\mathrm{id}}$, $\quad x^* \circ x = \sqrt{\dim(\theta)} 1_\theta$

$\dim(\theta)$= minimal dimension of θ.

Let $\mathcal{N} \overset{E}{\subset} \mathcal{M}$ with $\mathcal{M}$ in standard form in $B(\mathcal{H})$
i.e. $\exists \Phi \in \mathcal{H}$ cyclic and separating for $\mathcal{M}$.

Let $\Omega \in \mathcal{H}$ implement the state $(\Phi, E(\cdot)\Phi)$ on $\mathcal{M}$.

Define the Jones Projection:

$$e_{\mathcal{N}} := [\overline{\mathcal{N}\Omega}] \in \mathcal{N}'$$

implements expectation E: $e_{\mathcal{N}} m e_{\mathcal{N}} = E(m)\mathrm{e}_{\mathcal{N}}$ for $m \in \mathcal{M}$

Jones extension: $\mathcal{M}_1 := \mathcal{M} \vee \{e_{\mathcal{N}}\}$

$$\mathcal{N} \subset \mathcal{M} \subset \mathcal{M}_1$$

$\mathcal{N} \overset{E}{\subset} \mathcal{M}$ with $\mathcal{N}, \mathcal{M}$ properly infinite.

Let $\Phi \in \mathcal{H}$ be a bicyclic and biseparating vector for $\mathcal{N}, \mathcal{M}$.

Longo's canonical endomorphism:

$$\gamma : \mathcal{M} \to \mathcal{N}$$

$$\gamma(m) = J_{\mathcal{N},\Phi} J_{\mathcal{M},\Phi} m J_{\mathcal{M},\Phi} J_{\mathcal{N},\Phi}$$

$J_{\mathcal{N},\Phi}, J_{\mathcal{M},\Phi}$ are Tomita's modular conjugations of $\mathcal{N}$, $\mathcal{M}$ wrt Φ.

$$\gamma(\mathcal{M}) \subset \mathcal{N} \subset \mathcal{M} \subset \mathcal{M}_1$$

$\theta := \gamma_{\restriction \mathcal{N}}$ is canonical endomorphism for dual inclusion $\gamma(\mathcal{M}) \subset \mathcal{N}$

$\rightsquigarrow \theta$ is the dual canonical endomorphism of $\mathcal{N} \subset \mathcal{M}$

$$(\gamma(\mathcal{M}) \subset \mathcal{N}) \cong (\mathcal{M} \subset \mathcal{M}_1)$$

Definition

A Pimsner-Popa basis for $\mathcal{N} \overset{E}{\subset} \mathcal{M}$ is a collection $\{M_i\} \subset \mathcal{M}$, $i \in I$, such that

- $E(M_i M_j^*) = \delta_{i,j} q_i$, $\quad q_i$ projection in $\mathcal{N}$ $\quad$ (orthogonality "$\langle \xi_i | \xi_j \rangle = \delta_{i,j}$")
- $\sum_{i \in I} M_i^* e_{\mathcal{N}} M_i = \mathbb{1}$ $\quad$ (completeness "$\sum_i |\xi_i\rangle\langle\xi_i| = \mathbb{1}$")

$e_{\mathcal{N}}$ = Jones projection for $\mathcal{N} \overset{E}{\subset} \mathcal{M}$.

Expansion of $m \in \mathcal{M}$ wrt PiPo basis $\{M_i\} \subset \mathcal{M}$

$$m = \sum_{i \in I} M_i^* E(M_i m)$$

Convergence in the topology induced by E-invariant states.

Theorem (Fidaleo, Isola 99)

Every inclusion $\mathcal{N} \overset{E}{\subset} \mathcal{M}$ of properly infinite von Neumann algebras with a normal faithful conditional expectation $E : \mathcal{M} \to \mathcal{N}$ admits a Pimsner-Popa basis $\{M_i\} \subset \mathcal{M}$.

Definition (Fidaleo, Isola 99)

$$\theta \in \mathrm{End}(\mathcal{N}), \quad w \in \mathcal{N}, \quad \{m_i\} \subset \mathcal{N}$$

is a generalized (semidiscrete) Q-system iff

- $w^* w = \mathbb{1}$, $w : \mathrm{id} \to \theta$ in $\mathcal{N}$ ("intertwining property" of w)
- $m_i^* w w^* m_i$, $i \in I$ are mutually orthogonal projections in $\mathcal{N}$ such that $\sum_i m_i^* w w^* m_i = \mathbb{1}$ ("Pimsner-Popa condition")
- if $n \in \theta(\mathcal{N}) \vee \{m_i\}$ and $nw = 0$ then $n = 0$ ("faithfulness condition")

Theorem (Fidaleo, Isola 99)

Let $\mathcal{N}$ be a properly infinite von Neumann algebra with separable predual and $\theta \in \mathrm{End}(\mathcal{N})$. Then the following are equivalent

- *There is a von Neumann algebra $\mathcal{M}$ such that $\mathcal{N} \subset \mathcal{M}$ with $E \in E(\mathcal{M},\mathcal{N}) \neq \emptyset$, and θ is a dual canonical endomorphism for $\mathcal{N} \subset \mathcal{M}$, i.e., $\theta = \gamma_{\restriction \mathcal{N}}$ where $\gamma \in \mathrm{End}(\mathcal{M})$ is a canonical endomorphism for $\mathcal{N} \subset \mathcal{M}$.*
- *The endomorphism θ is part of a generalized Q-system, $(\theta, w, \{m_i\})$.*

Idea: Given $\mathcal{N} \overset{E}{\subset} \mathcal{M}$, choose γ canonical endomorphism.
Set $\theta = \gamma_{\restriction \mathcal{N}}$.
$E(\cdot) = w^*\gamma(\cdot)w$ for some $w : id \to \theta$.
Choose PiPo basis $\{M_i\}$ for $\mathcal{N} \overset{E}{\subset} \mathcal{M}$, and set $m_i := \gamma(M_i)$

Goal: Encode information of a net extension $\{\mathcal{A} \overset{E}{\subset} \mathcal{B}\}$ in data of $\{\mathcal{A}\}$.

Problem: Not clear how to "transport" generalized Q-systems to different local algebras.

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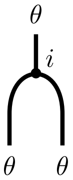
Problem: Not clear how to "transport" generalized Q-systems to different local algebras.

$\rightsquigarrow$ Restrict to a more specialized case.

Definition

A generalized Q-system of intertwiners in $\mathrm{End}(\mathcal{N})$ is a generalized Q-system $(\theta, w, \{m_i\})$ with additionally $m_i : \theta \to \theta^2$, $i \in I$.

$w =$ 

, $m_i =$ 

, $i \in I$

Turns out: gen Q-sys of intertwiners characterize discrete inclusions.

Definition

$\mathcal{N} \subset \mathcal{M}$ is discrete iff it is semidiscrete (i.e. $\exists E \in E(\mathcal{M}, \mathcal{N})$ normal, faithful conditional expectation) and $\hat{E}$ restricted to $\mathcal{M}_1 \cap \mathcal{N}'$ is semifinite.

Here $\hat{E} : \mathcal{M}_1 \to \mathcal{M}$ dual operator-valued *weight* ("unbounded expectation")

$$\mathcal{N} \overset{E}{\subset} \mathcal{M} \overset{\hat{E}}{\subset} \mathcal{M}_1 = \mathcal{M} \vee \{e_{\mathcal{N}}\}.$$

Proposition (Izumi, Longo, Popa 98)

If $\mathcal{N} \subset \mathcal{M}$ is an irreducible subfactor ($\mathcal{M} \cap \mathcal{N}' = \mathbb{C}$) then

$\mathcal{N} \subset \mathcal{M}$ discrete $\Leftrightarrow \theta \cong \oplus_i \rho_i$ with $\dim(\rho_i) < \infty$.

Theorem (S.D., Giorgetti)

Let $\mathcal{N}$ be an infinite factor with separable predual and $\theta \in \mathrm{End}(\mathcal{N})$. Then the following are equivalent

- *There is a von Neumann algebra $\mathcal{M}$ such that $\mathcal{N} \subset \mathcal{M}$ is discrete and θ is a dual canonical endomorphism for $\mathcal{N} \subset \mathcal{M}$.*
- *The endomorphism θ is part of a generalized Q-system of intertwiners in $\mathrm{End}(\mathcal{N})$, $(\theta, w, \{m_i\})$.*

Examples of (infinite index) discrete extensions in QFT are:

- [Doplicher, Roberts 90] canonical field net extensions $\{\mathcal{A} \subset \mathcal{F}\}$ in 3+1D with compact gauge group G, $\mathcal{A} = \mathcal{F}^G$.
- [Buchholz, Mack, Todorov 88] extensions $\{\mathcal{A} \subset \mathcal{B}\}$ in 1D with $\mathcal{A} = U(1)$-current, $G = \mathbb{T}$.
- Many extensions of $\{\mathrm{Vir}_{c=1}\}$, classified by [Carpi 04], [Xu 05].
- "Braided product" of discrete extensions.

Gen Q-sys of intertwiners $(\theta, w, \{m_i\})$ in $\mathrm{DHR}\{\mathcal{A}\}$: same thing with $\theta \in \mathrm{DHR}\{\mathcal{A}\}$.

Take θ localized in some reference region $\mathcal{O}$.
"Faithfulness condition" required for all spacetime regions $\tilde{\mathcal{O}} \supset \mathcal{O}$.

Theorem (S.D., Giorgetti)

Let $\{\mathcal{A}\}$ be a local net of infinite von Neumann factors fulfilling Haag duality and standardly realized on $\mathcal{H}_0$.

A generalized Q-system of intertwiners $(\theta, w, \{m_i\})$ in $\mathrm{DHR}\{\mathcal{A}\}$ induces an isotonous net of von Neumann algebras $\{\mathcal{B}\}$ such that $\{\mathcal{A} \subset \mathcal{B}\}$ is a standard, discrete inclusion of nets with a normal faithful standard conditional expectation E.

θ is the dual canonical endomorphism for $\{\mathcal{A} \subset \mathcal{B}\}$, $E(\cdot) = w^*\gamma(\cdot)w$ and "$\gamma^{-1}(m_i)$" is a Pimsner-Popa basis for $\mathcal{A}(\tilde{\mathcal{O}}) \overset{E}{\subset} \mathcal{B}(\tilde{\mathcal{O}})$, for all $\tilde{\mathcal{O}} \supset \mathcal{O}$.

Let $\{\mathcal{A} \subset \mathcal{B}^L\}$ and $\{\mathcal{A} \subset \mathcal{B}^R\}$ be extensions with gen Q-systems of intertwiners $(\theta^L, w^L, \{m_i^L\})$ and $(\theta^R, w^R, \{m_j^R\})$ in $\mathrm{DHR}\{\mathcal{A}\}$ (C^* *braided* tensor category). Then

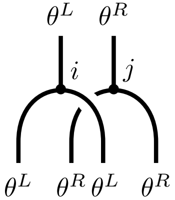
$$\big(\,\theta^L\theta^R,\, w^L w^R,\, \{m_i^L \times_\epsilon^\pm m_j^R\}\,\big)$$

where

$$m_i^L \times_\epsilon^\pm m_j^R := \theta^L(\epsilon^\pm_{\theta^L,\theta^R})\, m_i^L\, \theta^L(m_j^R)$$

is the "braided product" of two gen Q-sys of intertwiners.

$w^L w^R =$ 

, $m_i^L \times_\epsilon^+ m_j^R =$ 

, $(i,j) \in I \times J$

Theorem (S.D., Giorgetti)

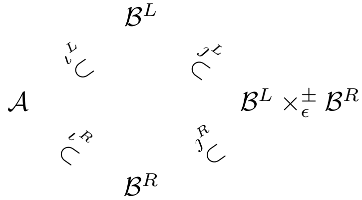
The braided product of two generalized Q-systems of intertwiners is again a generalized Q-system of intertwiners.

Braided Products

Theorem (S.D., Giorgetti)

The braided product of two generalized Q-systems of intertwiners is again a generalized Q-system of intertwiners.

$\rightsquigarrow$ construct a braided product extension $\{\mathcal{A} \subset \mathcal{B}^L \times^{\pm}_{\epsilon} \mathcal{B}^R\}$



- $\jmath^R \circ \iota^R = \jmath^L \circ \iota^L$
- $\jmath^L(\mathcal{B}^L) \vee \jmath^R(\mathcal{B}^R) = \mathcal{B}^L \times^{\pm}_{\epsilon} \mathcal{B}^R$
- $\jmath^R(M^R_j)\jmath^L(M^L_i) = \jmath^R \circ \iota^R(\epsilon^{\pm}_{\theta^L,\theta^R})\jmath^L(M^L_i)\jmath^R(M^R_j) \rightsquigarrow$ "One sided locality"

A transmissive boundary between a QFT $\mathcal{B}^L$ on the left halfspace and a QFT $\mathcal{B}^R$ on the right halfspace preserves energy and momentum.

Think of: $\mathcal{A}$ common stress-energy tensor for $\mathcal{B}^L$ and $\mathcal{B}^R$, not influenced by the boundary, $\{\mathcal{A} \subset \mathcal{B}^L\}$, $\{\mathcal{A} \subset \mathcal{B}^R\}$.

Locality for the observables in their own halfspace only requires that $\mathcal{B}^L(\mathcal{O}_1)$ commutes with $\mathcal{B}^R(\mathcal{O}_2)$ when $\mathcal{O}_1$ is in the left causal complement of $\mathcal{O}_2$ ("One sided locality").

Finite index case: classification of transmissive (irreducible) boundary conditions by central decomposition of braided product. [Bischoff, Kawahigashi, Longo, Rehren 16]. Irreducible boundary conditions are in correspondence with minimal central projections of braided product.

Discrete case: Open question. Not true that all irreducible boundary conditions are representations of braided product.

Example

Braided product of BMT local extensions of $U(1)$-current, $\{\mathcal{A}_{U(1)} \subset \mathcal{B}_\rho\}$. [Buchholz, Mack, Todorov 88]

$$\mathcal{Z}(\mathcal{B}_\rho \times_\epsilon^\pm \mathcal{B}_\rho) \cong L^\infty(\mathbb{S}^1, \mathrm{d}\mu)$$

In any case: central decomposition of $\mathcal{B}^L \times_\epsilon^\pm \mathcal{B}^R$ yields families of irreducible transmissive boundary conditions.

- Classification of transparent boundary conditions by central decomposition of braided product? (Discrete case)
- Construction of Longo-Rehren subfactor for discrete inclusions?
- Is it possible to improve "faithfulness condition" and express gen Q-sys of intertwiners as data in a W^* category?
- How to describe local net extensions in semidiscrete case?