Pseudodifferential calculus and Hadamard states

Local Quantum Physics and beyond
- in memoriam Rudolf Haag

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Christian Gérard
Départment of Mathématiques
Université Paris-Sud

The mode decomposition revisited

Cosmological spacetimes: $M = \mathbb{R}_t \times \Sigma$, (Σ, h) 3−dimensional Riemannian manifold, $g = -dt^2 + \lambda^2(t) h_{ij}(x) dx^i dx^j$.

Klein-Gordon operator: $-\Box_g + m^2 = \partial_t^2 + 3\frac{\dot{\lambda}}{\lambda}\partial_t - \lambda^{-2}\Delta_h + m^2$,

- $P := \lambda^{3/2}(-\Box_g + m^2)\lambda^{3/2} = \partial_t^2 + a(t,\epsilon)$, for $\epsilon = (-\Delta_h)^{\frac{1}{2}}$.

-the conserved charge for solutions of $P\phi = 0$ is $(\bar{\phi}_1 | q\phi_2) = \mathrm{i}(\partial_t \phi_1 | \phi_2)_{\mathcal{H}} - \mathrm{i}(\phi_1 \; \partial_t \phi_2)_{\mathcal{H}}$, for

$(u|v)_{\mathcal{H}} = \int_\Sigma \bar{u} v dVol_h$.

- Mode decomposition: associate 'creation-annihilation operators' to families of solutions of $P\phi = 0$ (see eg [Birrell-Davies]).

Mode decomposition

In modern language: creation-annihilation operators become a pure quasi-free state for the quantum Klein-Gordon field.

- assume that Σ is compact, $\epsilon = \sum_{j\in\mathbb{N}} \epsilon_j |e_j)(e_j|$.

$\phi_j(t,x) = \chi_j(t)e_j(x) \Rightarrow$ family of $1-d$ Schroedinger equations:

$\chi_j''(t) + a(t,\epsilon_j)\chi_j(t) = 0.$

One imposes the conditions:
$(\phi_j|q\phi_k) = -(\bar{\phi}_j|q\bar{\phi}_k) = \delta_{jk}$, $(\phi_j|q\bar{\phi}_k) = 0$ equivalent to

$\mathrm{i}\bar{\chi}_j'(0)\chi_j(0) - \mathrm{i}\bar{\chi}_j(0)\chi_j'(0) = 1.$

Heuristics: such a family produces a Hadamard state if (BKW)

$\chi_j(t) \sim_{j\to\infty} \mathrm{e}^{\mathrm{i}\int_0^t \sqrt{a(s,\epsilon_j)}ds}(a(t,\epsilon_j)^{-1/4} + \sum_{n\geq 1} c_n(t,\epsilon_j)\epsilon_j^{-n}).$

Mode decomposition

- more compact notation using functional calculus for ϵ: set

$\phi^{+}(t,\epsilon) = \sum_{j\in\mathbb{N}} \chi_j(t)|e_j)(e_j|$,

$\phi^{-}(t,\epsilon) = \sum_{j\in\mathbb{N}} \overline{\chi}_j(t)|e_j)(e_j| = \phi^{+}(t,\epsilon)^*$.

$T(\epsilon) = \begin{pmatrix} \phi^{+}(0,\epsilon) & \phi^{-}(0,\epsilon) \\ \mathrm{i}^{-1}\partial_t\phi^{+}(0,\epsilon) & \mathrm{i}^{-1}\partial_t\phi^{-}(0,\epsilon) \end{pmatrix}$. Then:

$T(\epsilon)^* q T(\epsilon) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

$\pi^{+} := T(\epsilon)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T(\epsilon)^{-1}$, $\pi^{-} := T(\epsilon)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} T(\epsilon)^{-1}$ are projections with

1) $\pi^{+} + \pi^{-} = \mathbb{1}$,
2) $(\pi^{+})^* q \pi^{-} = 0$ (ranges are $q-$orthogonal),
3) $\lambda^{\pm} := \pm q \circ \pi^{\pm} \geq 0$ ($\pm\pi^{\pm}$ is $q-$positive).

Quasi-free state associated to a mode decomposition

Consider the charged symplectic space $(\mathcal{H} \oplus \mathcal{H}, q)$, $\mathcal{H} = L^2(\Sigma)$.

We associate to it the CCR algebra generated by symbols $\psi(f)$, $\psi^*(f)$, $f \in \mathcal{H} \oplus \mathcal{H}$ with relations:

- $f \mapsto \psi^*(f)$ resp. $\psi(f)$ is $\mathbb{C}-$linear resp. antilinear.

- $[\psi(f_1), \psi(f_2)] = [\psi^*(f_1), \psi^*(f_2)] = 0$, $[\psi(f_1), \psi^*(f_2)] = \bar{f}_1 \cdot qf_2 \mathbb{1}$,

-$\psi(f)^* = \psi^*(f)$.

There is a unique quasi-free state ω on $CCR(\mathcal{H} \oplus \mathcal{H}, q)$ defined by:

$\omega(\psi(f_1)\psi^*(f_2)) = \bar{f}_1 \cdot \lambda^+ f_2$,

$\omega(\psi^*(f_2)\psi(f_1)) = \bar{f}_1 \cdot \lambda^- f_2$.

note $\lambda^\pm \geq 0$, $\lambda^+ - l^- = q$.

Quasi-free state associated to a mode decomposition

ω induces a quasi-free state for the quantum Klein-Gordon field on M by the isomorphisms:

$$E : (\frac{C_0^\infty(M)}{PC_0^\infty(M)}, \mathrm{i}^{-1}E) \to (\mathrm{Sol}_{\mathrm{sc}}(P), q)$$

E causal propagator,

$$\begin{array}{ll} \rho : & (\mathrm{Sol}_{\mathrm{sc}}(P), q)(C_0^\infty(\Sigma) \oplus C_0^\infty(\Sigma), q) \\ & \phi \mapsto (\phi_{|t=0}, \mathrm{i}^{-1}\partial_t\phi_{|t=0}), \ \textit{Cauchy data map}. \end{array}$$

Quasi-free state associated to a mode decomposition

A side question: how to justify that WKB solutions produce Hadamard states:

1) if $\Sigma = \mathbb{R}^3$ or $\mathbb{S}^3$ with their standard metrics: use Fourier analysis.

2) if (Σ, h) arbitrary complete Riemannian manifold: one needs to use more advanced tools:
pdo calculus, Egorov's theorem, see later.

The mode decomposition method, although limited to cosmological models, allows to understand many things:

1) non-uniqueness of Hadamard states: different solutions have same WKB expansion,

2) adiabatic vacua: stop the expansion after a finite number of terms.

Pdo calculus on manifolds

We start with pseudodifferential calculus on $\mathbb{R}^d$:

symbol classes: $a(x,\xi) \in S^m(T^*\mathbb{R}^d)$ for $m \in \mathbb{R}$ if

$\partial_x^\alpha \partial_\xi^\beta a(x,\xi) \in O(\langle\xi\rangle^{m-|\beta|})$, $\alpha, \beta \in \mathbb{N}^d$.

quantization of symbols: $A \in \Psi^m(\mathbb{R}^d)$ if

$Au(x) = \mathrm{Op}(a)u()x = (2\pi)^{-d} \int \mathrm{e}^{\mathrm{i}(x-y)\cdot\xi} a(x,\xi)u(y)dyd\xi$,
$u \in C_0^\infty(\mathbb{R}^d)$.

- A preserves $\mathcal{S}(\mathbb{R}^d)$,
- $\Psi^\infty(\mathbb{R}^d) = \bigcup_{m\in\mathbb{R}} \Psi^m(\mathbb{R}^d)$ is a graded $*-$algebra.

Pdo calculus on manifolds

Let Σ a smooth manifold.
a linear operator $A : C_0^\infty(\Sigma) \to C_0^\infty(\Sigma)$ belongs to $\Psi_{\rm c}^m(\Sigma)$ if:

1) A is properly supported: $\pi_x, \pi_y : {\rm supp} A(\cdot, \cdot) \to \Sigma$ is proper,

2) if $U_1, U_2 \subset \Sigma$ are chart neighborhoods, $\psi_i : U_i \to \mathbb{R}^d$ chart diffeomorphisms, $\chi_i \in C_0^\infty(U_i)$ then:

$$\chi_1 \circ A \circ \chi_2 = \psi_2^* \circ B \circ (\psi_1^*)^{-1}, \text{ for } B \in \Psi^m(\mathbb{R}^d).$$

$\Psi_{\rm c}^\infty(\Sigma)$ is a graded $*-$algebra.

Well defined notion of principal symbol $\sigma_{\rm pr}(A)$.

Problem: if $A = a(x, \partial_x)$ elliptic, selfadjoint differential operator ($A = -\Delta_h$, for h complete Riemannian metric on Σ), then

$$A \in \Psi_{\rm c}^\infty(\Sigma), \text{ but } (A + {\rm i})^{-1} \not\in \Psi_{\rm c}^\infty(\Sigma).$$

$\Psi_{\rm c}^\infty(\Sigma)$ not closed under inverses !

Pdo calculus on manifolds

Instead one has

$$(A + \mathrm{i})^{-1} \in \Psi^{\infty}(\Sigma) = \Psi^{\infty}_{\mathrm{c}}(\Sigma) + \mathcal{W}^{-\infty}(\Sigma),$$

where $\mathcal{W}^{-\infty}(\Sigma)$ is the ideal of smoothing operators.

Not a good solution: $\Psi^{\infty}(\Sigma)$ is not an algebra: operators in $\Psi^{\infty}(\Sigma)$ cannot be composed!

Need for an intermediate calculus, located between $\Psi^{\infty}_{\mathrm{c}}(\Sigma)$ and $\Psi^{\infty}(\Sigma)$.

A convenient calculus is Shubin's calculus of uniform pdos $\Psi^{\infty}_{\mathrm{bg}}(\Sigma)$, relying on the notion of manifolds of bounded geometry.

$\Psi^{\infty}_{\mathrm{bg}}(\Sigma)$ is a graded $*-$algebra, stable under (elliptic) inverses.

A more general framework for Hadamard states

We fix (M, g) globally hyperbolic spacetime, $\Sigma \subset M$ space-like Cauchy surface.

$\rho_{\Sigma} : C^{\infty}_{\rm sc}(M) \ni \phi \mapsto (\phi_{|\Sigma}, {\rm i}^{-1}\partial_{\nu}\phi_{|\Sigma}) \in C_0^{\infty}(\Sigma) \otimes \mathbb{C}^2$.

normal Gaussian coordinates: $\chi : U \ni (s, x) \mapsto \exp^g_x(sn_x) \in V$
U neighb. of $\{0\} \times \Sigma$ in $\mathbb{R} \times \Sigma$,
V neighb. of Σ in M, n_x future unit normal at $x \in \Sigma$.

Standing hypothesis: $]-\epsilon, \epsilon[\times\Sigma \subset U$ for some $\epsilon > 0$.

We are reduced to $M = I \times \Sigma$, I open interval,

$g = -dt^2 + h_{ij}(t, x)dx^i dx^j$.

Space-time and Cauchy surface covariances

The KG equation $(-\Box_g + m)\phi = 0$ can be reduced to:

$\partial_t^2 \phi + a(t, x, \partial_x)\phi = 0,$

for $a(t, x, \partial_x)$ 2^{nd} order, elliptic, selfadjoint for $(u|v)_\Sigma = \int_\Sigma \bar{u} v |h_0|^{\frac{1}{2}} dx.$

The conserved charge is

$(\phi|q\phi) = \mathrm{i} \int_\Sigma (\partial_t \bar{\phi} \phi - \bar{\phi} \partial_t \phi) |h_0|^{\frac{1}{2}} dx.$

Space-time and Cauchy surface covariances

Let ω be a quasi-free state for quantum Klein-Gordon field.

Space-time covariances: $\Lambda^{\pm} : C_0^{\infty}(M) \to C^{\infty}(M)$

1) $P\Lambda^{\pm} = \Lambda^{\pm}P = 0$,

2) $\Lambda^{+} - \Lambda^{-} = \mathrm{i}^{-1}E$,

3) $(u|\Lambda^{\pm}u) \geq 0$, $u \in C_0^{\infty}(M)$.

Cauchy surface covariances: $\lambda_{\Sigma}^{\pm} : C_0^{\infty}(\Sigma) \otimes \mathbb{C}^2 \to C^{\infty}(\Sigma) \otimes \mathbb{C}^2$

1) $\lambda^{+} - \lambda^{-} = q$, $q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

2) $(f|\lambda_{\Sigma}^{\pm}f) \geq 0$, $f \in C_0^{\infty}(\Sigma) \otimes \mathbb{C}^2$.

Space-time and Cauchy surface covariances

Link between the two objects:

$\Lambda^{\pm} = (\rho \circ E)^{*} \lambda^{\pm}_{\Sigma} (\rho \circ E)$, ρ Cauchy data map

$\lambda^{\pm}_{\Sigma} = \pm q \circ \pi^{\pm}$ with

$$\pi^{\pm} = \begin{pmatrix} \mathrm{i}\partial_s \Lambda^{\pm}(0,0) & \Lambda^{\pm}(0,0) \\ \partial_t \partial_s \Lambda^{\pm}(0,0) & \mathrm{i}^{-1}\partial_t \Lambda^{\pm}(0,0) \end{pmatrix}$$

$\Lambda^{\pm}(t,s) : C_0^{\infty}(\Sigma) \to C^{\infty}(\Sigma)$ time-kernel of $\Lambda^{\pm}$.

Problem: conditions on $\lambda^{\pm}_{\Sigma}$ ensuring that ω is a pure Hadamard state?

1) purity: $\pi^{\pm}$ should be projections.

2) Hadamard property: if $f \in \mathcal{E}'(\Sigma) \otimes \mathbb{C}^2$ and $\pi^{\pm} f = f$ then $\mathrm{WF}(Uf) \subset \mathcal{N}^{\pm}$,
where Uf solution of the Cauchy problem for f, $\mathcal{N}^{\pm}$ positive/negative energy surfaces see [GW], [GOW].

Construction of Hadamard states

To ensure condition 2) we look for operators $b(t) = b(t, x, \partial_x) \in \Psi^1(\Sigma)$ such that:

$$(\partial_t^2 + a(t, x, \partial_x))\mathrm{Texp}(\mathrm{i}\int_0^t b(s)ds) = 0.$$

- equivalent to the following Riccati equation:

$$(R)\ \mathrm{i}\partial_t b(t) - b^2(t) + a(t) = 0.$$

Can be solved modulo $\Psi^{-\infty}(\Sigma)$ with the ansatz:

$$b(t) = a(t)^{\frac{1}{2}} + \sum_{j=0}^{\infty} b_{-j}(t),\ \ b_{-j}(t) \in \Psi^{-j}(\Sigma).$$

- Exact analog of WKB solutions in the cosmological case!

Construction of Hadamard states

If $b^+(t) = b(t)$ is a solution of (R), then $b^-(t) = -b^*(t)$ is another solution.

(R) is equivalent to a factorization:

$$\partial_t^2 + a(t) = (\partial_t + \mathrm{i}b(t)) \circ (\partial_t - \mathrm{i}b(t)) \text{ modulo } \Psi^{-\infty}(\Sigma).$$

Junker (1995) noticed the relevance of such a factorization to prove that a state is Hadamard.
Main result ([G-Wrochna], [G-Oulghazi-Wrochna]): set

$$T = \mathrm{i}^{-1} \begin{pmatrix} 1 & -1 \\ b^+(0) & -b^-(0) \end{pmatrix} (b^+(0) - b^-(0))^{-\frac{1}{2}}.$$

Then:

$$\lambda^+ = T^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T, \ \lambda^- = T^* \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} T$$

are the Cauchy surface covariances of a pure Hadamard state.

Construction of Hadamard states

To make the construction rigorous one needs a global pdo calculusø on Σ. In particular:

1) $(b^{+}(0) - b^{-}(0))^{-\frac{1}{2}}$ should also be a pdo (example of a Seeley's theorem on powers of elliptic pdos)

2) to show Hadamard property one needs an Egorov's theorem: if $A \in \Psi^{m}(\Sigma)$ then

$$A(t) = \mathrm{Texp}(\mathrm{i}\int_0^t b(s)ds) \circ A \circ \mathrm{Texp}(\mathrm{i}\int_t^0 b(s)ds) \in \Psi^{m}(\Sigma)$$

and $\sigma_{\rm pr}(A(t)) = \sigma_{\rm pr}(A) \circ \Phi_{0,t}$, $\Phi_{s,t}$ symplectic flow for the time-dependent Hamiltonian $\sigma_{\rm pr}(b)(t, x, \xi)$.

Construction of Hadamard states

First done in [GW] for $\Sigma = \mathbb{R}^d$ using the standard pdo calculus on $\mathbb{R}^d$

Extended in [GOW] to a much wider framework: (Σ, h) Riemannian manifold of bounded geometry: related notion of bounded geometry for Lorentzian manifolds.

Examples of applications:

1) perturbations of Kerr-Kruskal, exterior Kerr- de Sitter.

2) future/past lightcones, double cones, wedges in Minkowski.

The appropriate pdo calculus is Shubin's $\Psi_{\rm bg}(\Sigma)$ calculus: relies on Gaussian normal coordinates for a reference Riemannian metric + appropriate ideal of smoothing operators.

Hadamard states on arbitrary spacetimes

A consequence of a result in [GW] is as follows:

1) for any spacetime (M, g) globally hyperbolic and $\Sigma \subset M$ spacelike Cauchy surface there exists one Hadamard state ω such that $\lambda_\Sigma^\pm$ belong to $\Psi_{\rm c}^\infty(\Sigma)$. ($\omega$ is not pure !)

2) for all other Hadamard states $\lambda_\Sigma^\pm$ belong to $\Psi^\infty(\Sigma)$.

The states constructed in [GOW] lie in between $\Psi_{\rm c}^\infty(\Sigma)$ and $\Psi^\infty(\Sigma)$, ie in $\Psi_{\rm bg}^\infty(\Sigma)$.

Another application of pdo calculus: the Hartle-Hawking-Israel state

We consider the situation studied by Sanders [S] (first rigorous construction of the HHI state for static bifurcate Killing horizons):
Framework:

- (M, g) globally hyperbolic,

- V complete Killing vector field for (M, g),

- $\mathcal{B} = \{x \in M : V(x) = 0\}$ bifurcation surface: compact, connected orientable submanifold of codimension 2,

- there exists Σ spacelike Cauchy surface with $\mathcal{B} \subset \Sigma$,

- V is $g-$orthogonal to Σ (V is static).

- using the two null directions normal to $\mathcal{B}$ one generates a bifurcate Killing horizon $\mathcal{H} = \mathcal{H}_{\rm l} \cup \mathcal{H}_{\rm r}$.

- the scalar $\kappa > 0$ defined by $\kappa^2 = -\frac{1}{2}\nabla^a V^b \nabla_a V_b$ is constant on $\mathcal{H}$: surface gravity.

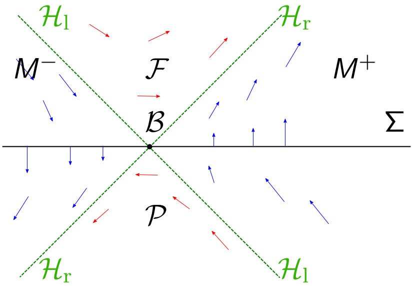

Figure: spacetime with bifurcate Killing horizon

$M^{\pm} = D(\Sigma^{\pm})$: right/left wedges, $\mathcal{F}, \mathcal{P} = I^{\pm}(\mathcal{B})$ future/past cones.
All are globally hyperbolic spacetimes.

bifurcate Killing horizons

An additional assumption is the existence of a wedge reflection:
- $R : (U, g) \to (U, g)$ isometry with $R \circ R = Id$, (U neighborhood of $M^+ \cup M^-$), reversing the time orientation,

- $R = Id$ on $\mathcal{B}$, $R^* V = V$ on $M^+ \cup M^-$.

One considers the free quantum Klein-Gordon field on (M, g) given by the Klein-Gordon equation:

$$(-\Box_g + m^2(x))\phi = 0, \ \ m(x) \geq m_0 > 0,$$

and m invariant under the Killing field V and wedge reflection R.

The double $\beta-$KMS state

Let ω_β^+ be the thermal state on (M^+, g) w.r.t. the time-like isometry group generated by V.

- Kay showed how to extend ω_β^+ to $M^+ \cup M^-$, using the wedge reflection R:

- one obtains the double $\beta-$KMS state ω_β, a pure Hadamard state on $(M^+ \cup M^-, g)$.

- ω_β is completely analogous to the vacuum vector in the Araki-Woods representation of a thermal state.

- Sanders (2013) proved that there exists a unique Hadamard extension $\omega_{\rm HHI}$ of ω_β to (M, g) if and only if:
$\beta^{-1} = T_{\rm H} = \frac{\kappa}{2\pi}$ Hawking temperature.

Construction by pdo calculus

Let t Killing time coordinate in M^{+}: $M^{+} \sim \mathbb{R} \times \Sigma$,

$g = -v^2(y)dt^2 + h_{ij}(y)dy^i dy^j$, $v^2 = -V^a V_a$.

Wick rotation: $t = \mathrm{i}\tau$ produces the Riemannian manifold $(N, \hat{g})$ for

$$N = \mathbb{S}_\beta \times \Sigma^{+}, \hat{g} = v^2(y)d\tau^2 + h_{ij}(y)dy^i dy^j.$$

- the associated Laplacian is

$$K = -\Delta_{\hat{g}} + m^2(y)$$

- we set $\Omega =]0, \beta/2[\times\Sigma^{+}$ open subset of N.

- $\partial\Omega$ has two connected components $S^0 = \{\tau = 0\}$ and $S^{\beta/2} = \{\tau = \beta/2\}$.

- S^0 identified with Σ^{+}, $S^{\beta/2}$ identified with $\Sigma^{-} = R(\Sigma^{+})$.

The Calderón projector

The Calderón projector is a standard object in elliptic boundary value problems:

if $u \in C^\infty(\overline{\Omega})$ and $\gamma u := \begin{pmatrix} u_{|\partial\Omega} \\ \partial_\nu u_{|\partial\Omega} \end{pmatrix}$, ∂_ν unit normal then the Calderón projector $D : C_0^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$ is defined as:

$$Df := \gamma \circ K^{-1}(\delta_{\partial\Omega} \otimes f_1 + \partial_\nu \delta_{\partial\Omega} \otimes f_0), \; f = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}.$$

A well-known result: D is given by a matrix of pdos on Σ.
Thm[G]: Let $\lambda_\beta^\pm$ be the Cauchy surface covariances of ω_β and $c^\pm = q^{-1} \circ \lambda_\beta^\pm$. Then

$$c_\beta^+ = D.$$

(valid for any $\beta > 0$). Both sides are operators on $\Sigma^+ \cup \Sigma^-$.

The Calderón projector

- If $\beta = T_H^{-1}$ then $(N, \hat{g})$ has a unique smooth extension $(N_{\rm ext}, \hat{g}_{\rm ext})$ (well-known fact):

$$\begin{aligned} \psi: \quad & \mathbb{S}_\beta \times \Sigma^+ \to \mathbb{R}^2 \times \mathcal{B} = N_{\rm ext} \\ & (\tau, s, \omega) \mapsto (s\cos(\frac{2\pi}{\beta}\tau), s\sin(\frac{2\pi}{\beta}\tau), \omega) \end{aligned}$$

for (s, ω) Gaussian normal coordinates to $\mathcal{B}$ in Σ.

For other values of β $\hat{g}_{\rm ext}$ has a conical singularity on $\mathcal{B}$.

- Moreover ψ restricts to a smooth embedding of Σ into $N_{\rm ext}$.

- Consequence: a natural candidate for the extension of c^+ to Σ is $D_{\rm ext}$, the Calderón projector for $K_{\rm ext} = -\Delta_{\hat{g}_{\rm ext}} + m^2$, associated to the open set $\psi(\Omega)$.

The HHI state

We set:

$$\lambda^{+}_{\rm HHI} = q \circ D_{\rm ext}, \ \lambda^{-}_{\rm HHI} = q \circ (\mathbb{1} - D_{\rm ext}).$$

Then:
1) $\lambda^{\pm}_{\rm HHI} \geq 0$, proof as in [S], uses reflection positivity of K and $K_{\rm ext}$,
2) $\lambda^{\pm}_{\rm HHI}$ are the unique extensions of $\lambda^{\pm}_{\beta}$ with the property that they map $C_0^{\infty}(\Sigma) \otimes \mathbb{C}^2$ into $C^{\infty}(\Sigma) \otimes \mathbb{C}^2$.
3) $\lambda^{\pm}_{\rm HHI}$ are Hadamard covariances.

an elementary proof of the Hadamard property

Sanders used the Hadamard parametrix construction to show that $\omega_{\rm HHI}$ is Hadamard.

Using pdo calculus one can give a rather elementary proof:

1) $\lambda^{+}_{\rm HHI}\in \Psi^{\infty}(\Sigma)$ since it is a Calderón projector.

2) pick a Hadamard state $\omega_{\rm ref}$ on M. One knows that $\lambda^{\pm}_{\rm ref}\in \Psi^{\infty}(\Sigma)$.

3) both states are Hadamard in $M^{+}\cup M^{-}$: by a well-known result of Radzikowski, this implies that

$\lambda^{+}_{\rm ref}-\lambda^{+}_{\rm HHI}$ 'smoothing in $\Sigma^{+}\cup \Sigma^{-}$' ie

$\chi\circ(\lambda^{+}_{\rm ref}-\lambda^{+}_{\rm HHI})\circ\chi$ is smoothing if ${\rm supp}\chi\cap \mathcal{B}=\emptyset$.

4) this implies that $\lambda^{+}_{\rm ref}-\lambda^{+}_{\rm HHI}$ is smoothing on the whole of Σ: look at the principal symbol and argue by continuity.