Sigma models in algebraic QFT

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partly joint work with

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Local Quantum Physics and Beyond

In Memoriam Rudolf Haag

Hamburg

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Algebraic QFT

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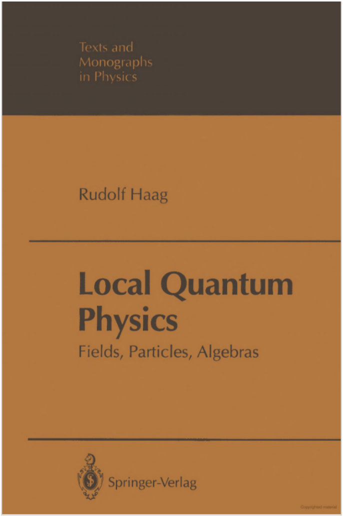
- Book “*Local Quantum Physics – Fields, Particles, Algebras*”
- Describes QFT via families of *local algebras*

$$\mathcal{O} \longmapsto \mathcal{A}(\mathcal{O})$$

instead of quantum fields



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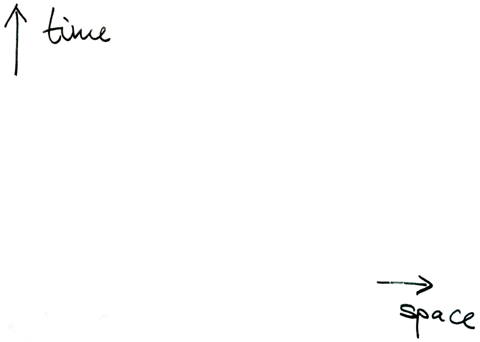

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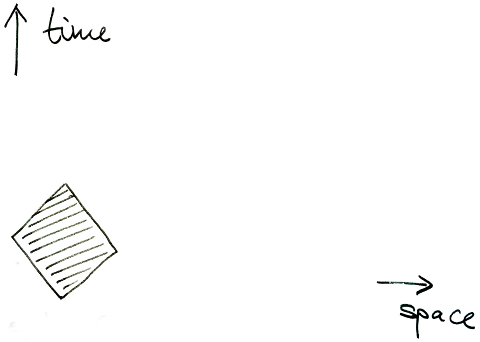

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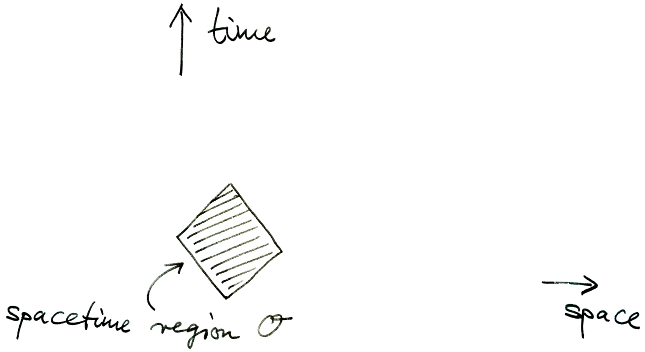

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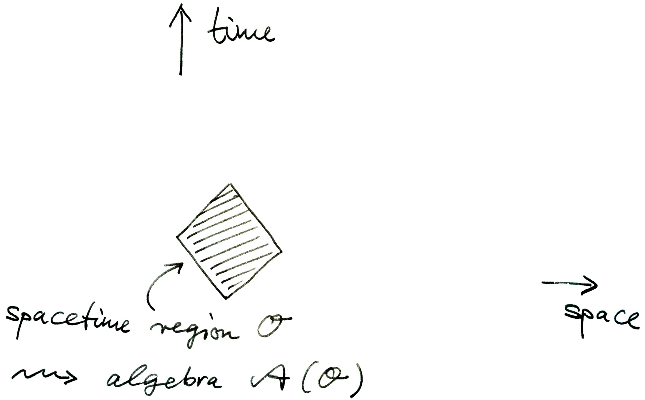

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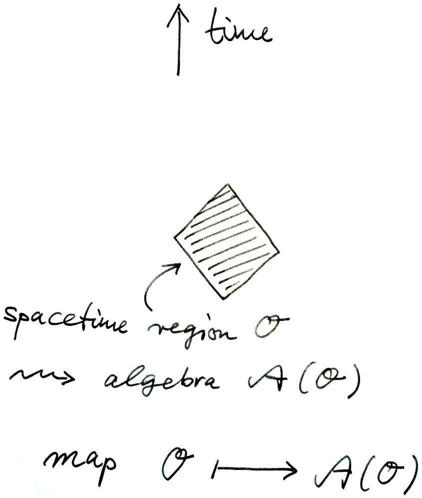

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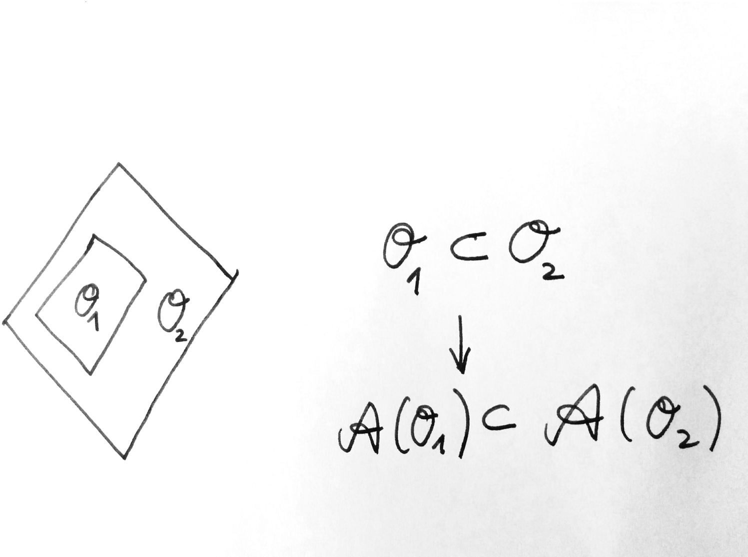

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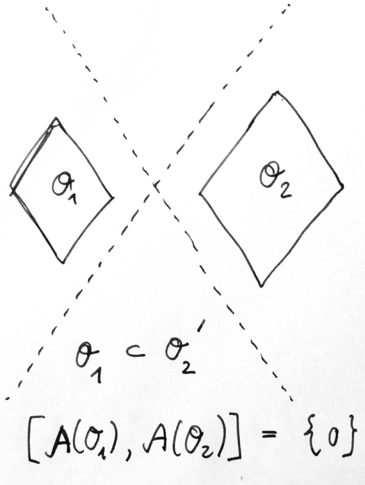

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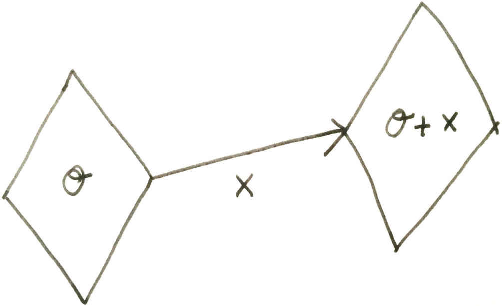

$$\mathcal{A}(\mathcal{O}+x) = \alpha_x(\mathcal{A}(\mathcal{O}))$$

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"Axioms":

- Isotony: Inclusions of regions give inclusions of algebras
- Locality: Algebras of spacelike separated regions commute
- Covariance: The isometry group of spacetime acts covariantly by automorphisms
- further axioms regarding states (vacuum ....)

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AQFT has led to deep conceptual insights. Examples:

- Doplicher-Haag-Roberts theory of localized charges / global gauge theories
- Haag-Ruelle scattering theory
- Formulation of thermal equilibrium (KMS) states [Haag, Hugenholtz, Winnink '67]

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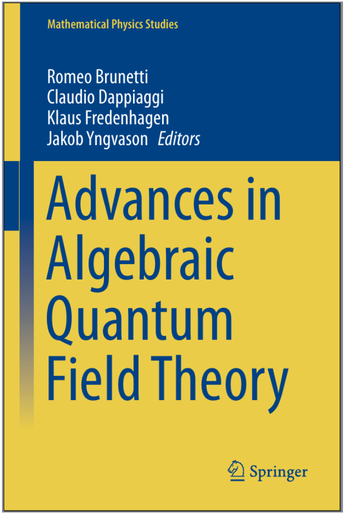

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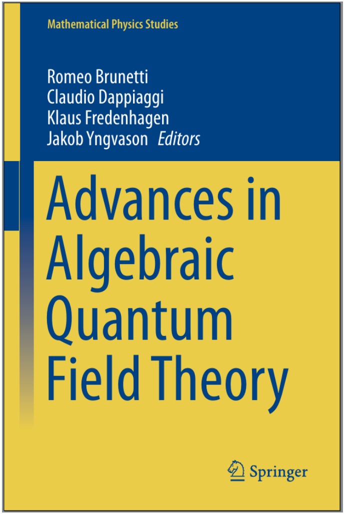
There now exist several programmes aiming at building models with the tools of AQFT:

- perturbative AQFT
  $\rightarrow$ *talk by Rejzner*
- conformal AQFT
  $\rightarrow$ *talk by Longo*
- AQFT on curved or quantum spacetimes
  $\rightarrow$ *talks by Doplicher, Gérard*
- low-dimensional AQFT models
  $\rightarrow$ *this talk*



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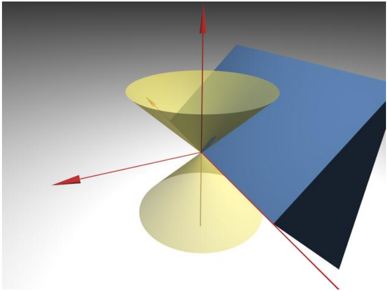

- $\Lambda(t)$: boosts into W
- $\Delta^{it} := U_1(\Lambda(-2\pi t))$
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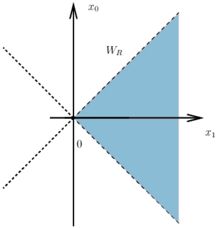

Gives (real) spaces $H(W) \subset \mathcal{H}_1$ of “localized vectors”.

Example: $d = 1+1$, massive irreducible rep U_1, gauge group arbitrary. In this case, we have

$$\mathcal{H}_1 = L^2(\mathbb{R}, d\theta) \otimes \mathcal{K}, \qquad (\Delta^{it}\psi)(\theta) = \psi(\theta - 2\pi t), \qquad (\mathcal{J}_1\psi)(\theta) = \Gamma_1\psi(\theta)$$

and $H(W)$ = Hardy space $H^2 \otimes \mathcal{K} \subset L^2 \otimes \mathcal{K}$ on the strip $0 < \mathrm{Im}(\theta) < \pi$ satisfying

$$h(\theta + i\pi) = \Gamma_1 h(\theta), \qquad \theta \in \mathbb{R}.$$

- Localization in smaller regions = simultaneous localization in several wedges:
$$H(\bigcap_i W_i) = \bigcap_i H(W_i).$$
- If U_1 has positive energy, this always leads to a meaningful concept of localization of vectors [Brunetti, Guido, Longo].
- A free QFT (with localized *algebras*) follows by second quantization:
$$\mathcal{A}(\mathcal{O}) = \{e^{i(a^\dagger(h)+a(h))} : h \in H(\mathcal{O})\}'', \qquad a, a^\dagger : CCR.$$

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Free QFT is the basis for constructions of *interacting* QFTs.

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Simplest deformation of CCR over $\mathcal{H}_1 = L^2(\mathbb{R}, d\theta) \otimes \mathbb{C}^N$:

$$a_\mu(\theta)a_\nu(\theta') = a_\nu(\theta')a_\mu(\theta)$$
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- $\mu, \nu, \ldots$: labels basis in $\mathcal{K} = \mathbb{C}^N$
- $R(\theta)$: Unitary map $\mathcal{K} \otimes \mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$

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More general deformations conceivable, but this is the easiest case

Invariant Yang-Baxter operators

- Associativity of algebra of the $a, a^\dagger$ requires R to solve the Yang-Baxter equation (on $\mathcal{K} \otimes \mathcal{K} \otimes \mathcal{K}$):

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Further requirements on R:

- R must be Poincaré invariant and gauge invariant (commute with $U_1 \otimes U_1$ and $V_1 \otimes V_1$), including TCP invariance.
- R must be *crossing symmetric*: $\theta \mapsto R(\theta)$ analytically extends to $0 < \mathrm{Im}(\theta) < \pi$ and

$$\langle \xi \otimes \psi,\, R(i\pi - \theta)\,(\varphi \otimes \xi') \rangle_{\mathcal{K}\otimes\mathcal{K}} = \langle \psi \otimes \Gamma_1 \xi',\, R(\theta)\,(\Gamma_1 \xi \otimes \varphi) \rangle_{\mathcal{K}\otimes\mathcal{K}} .$$

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Will mainly focus on two examples: $G = O(N)$ and $G = O(N, 1)$

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These will lead to $O(N)$- and $O(N,1)$- Sigma models

Vacuum representations and wedge-locality

We define a *vacuum state* ω on the algebra CCR_R of the "deformed" creation/annihilation operators:

$$\omega(\cdots a_R) = 0 = \omega(a_R^\dagger \cdots), \qquad \omega(1) = 1\,.$$

Theorem

Let R be a crossing-symmetric G-invariant Yang-Baxter operator, and let $(\pi, \mathcal{H}, \Omega, U, V)$ be the GNS representation of CCR_R w.r.t. ω. Then the von Neumann algebra

$$\mathcal{M}_R := \{e^{i(\pi(a_R^\dagger(h)+a_R(h))} \;:\; h \in H(W)\}'' \subset \mathcal{B}(\mathcal{H})$$

is localized in the wedge W in the sense that

- $U(x,\lambda)\mathcal{M}_R U(x,\lambda)^{-1} \subset \mathcal{M}_R$ *for* $x \in W$
- $\mathcal{J}\mathcal{M}_R\mathcal{J} = \mathcal{M}_R'$
- *Vacuum* Ω *is cyclic and separating for* $\mathcal{M}_R$

Any such wedge algebra is a germ of a full QFT.

Vacuum representations and wedge-locality

We define a *vacuum state* ω on the algebra CCR_R of the "deformed" creation/annihilation operators:

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The $O(N)$ model

Take $G = O(N)$ in its defining representation on $\mathbb{C}^N$.

$R(\theta) : \mathbb{C}^N \otimes \mathbb{C}^N \to \mathbb{C}^N \otimes \mathbb{C}^N$ is essentially fixed by invariance, YBE and crossing:

$$R(\theta) = \sigma_1(\theta)\, Q + \sigma_2(\theta)\, 1 + \sigma_3(\theta)\, F$$

with F =tensor flip, Q a 1-dim $O(N)$-invariant projection, and

$$\sigma_2(\theta) = \frac{\Gamma\left(\frac{1}{N-2} - i\frac{\theta}{2\pi}\right)\Gamma\left(\frac{1}{2} - i\frac{\theta}{2\pi}\right)\Gamma\left(\frac{1}{2} + \frac{1}{N-2} + i\frac{\theta}{2\pi}\right)\Gamma\left(1 + i\frac{\theta}{2\pi}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{N-2} - i\frac{\theta}{2\pi}\right)\Gamma\left(-i\frac{\theta}{2\pi}\right)\Gamma\left(1 + \frac{1}{N-2} + i\frac{\theta}{2\pi}\right)\Gamma\left(\frac{1}{2} + i\frac{\theta}{2\pi}\right)},$$

$$\sigma_1(\theta) = -\frac{2\pi i}{N-2} \cdot \frac{\sigma_2(\theta)}{i\pi - \theta},$$

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This is Zamolodchikov's $O(N)$-invariant two-particle S-matrix.

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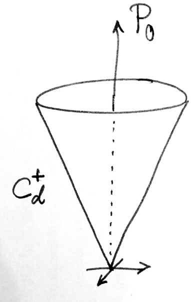
The $O(d,1)$ model

Take $G = O(d,1)$ = Lorentz group in $d+1$ dimensions in a principal or complementary series irrep V_1 on $\mathcal{K}$.

- $\mathcal{K}$ can be realized as a space of homogeneous functions on a light cone

$$C_d^+ = \{P \in \mathbb{R}^{d+1} \ : \ P \cdot P = 0 \quad P_0 > 0\}\,,$$

$$\mathcal{K}_\nu = \{\psi : C_d^+ \to \mathbb{C} \ : \ \psi\,(\lambda \cdot P) = \lambda^{-\frac{d-1}{2} - i\nu} \cdot \psi\,(P)\,, \lambda > 0\}$$



ν: complex parameter.

- $SO_+(d,1)$ acts by $(V_\nu(\Lambda)\psi)(P) = \psi(\Lambda^{-1}P)$.
  - For certain ν, find scalar product on $\mathcal{K}_\nu$ such that V_ν is unitary:
    - principal series: $\nu \in \mathbb{R}$.
    - complementary series: $i\nu \in (0, \frac{d-1}{2})$
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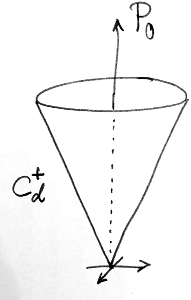

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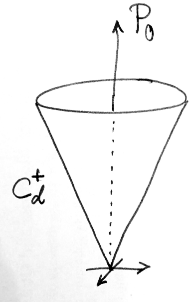

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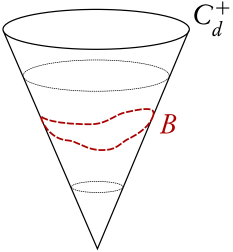
Some representation theory of $SO_+(d,1)$

- To define a scalar product on $\mathcal{K}_\nu$, pick an "orbital base" B and the $(d-1)$-form

$$\omega = \sum_{k=1}^{d} (-1)^{k+1} \frac{P_k}{P_0} \, dP_1 \wedge \ldots \wedge \widehat{dP_k} \wedge \ldots \wedge dP_d \, .$$

For principal series, define inner product $(\psi_1, \psi_2)_\nu := \int_B \omega \, \overline{\psi_1} \, \psi_2$. —> makes V_ν unitary.



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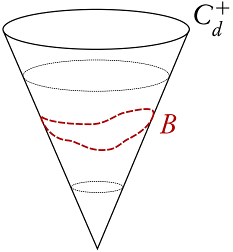
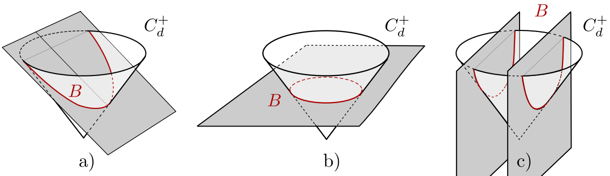


Three canonical choices for B:

a) **flat base**, $B = C_d^+ \cap$ (lightlike plane)

b) **spherical base**, $B = C_d^+ \cap$ (spacelike plane)

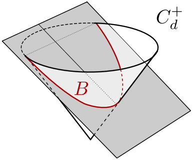
c) **hyperbolic base**, $B = C_d^+ \cap$ (two parallel timelike planes)



a) b) c)

The flat base and Euclidean conformal symmetry

- Take complementary series rep $\nu \in i(0, \frac{d-1}{2})$ and "flat base" B of C_d^+



- B parameterized as $\mathbb{R}^{d-1} \ni \boldsymbol{x} \mapsto P(\boldsymbol{x}) = \left(\frac{1}{2}(|\boldsymbol{x}|^2+1), \boldsymbol{x}, \frac{1}{2}(|\boldsymbol{x}|^2-1)\right)$
- Representation space $\mathcal{K}_\nu$ has scalar product

$$(f, g) = c_\nu \int d^d\boldsymbol{x} \int d^d\boldsymbol{y}\, \overline{f(\boldsymbol{x})}\, |\boldsymbol{x}-\boldsymbol{y}|^{-2s}\, g(\boldsymbol{y})$$

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$$(V_\nu(\Lambda)f)(\boldsymbol{x}) = Y_\Lambda(\boldsymbol{x})^{-\frac{d-1}{2}-i\nu} \cdot f(\Lambda \cdot \boldsymbol{x})$$

This double role of SO$(d,1)$ is at the basis of the dS/CFT correspondence.

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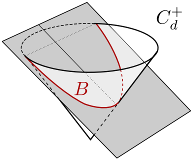

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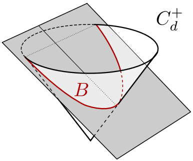

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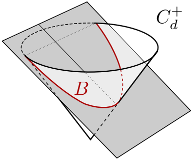

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$SO_+(d,1)$-invariant R-matrices

An "R-matrix" for the representations V_ν of $SO(d,1)$ is an integral operator

$$R(\theta) : \mathcal{K}_\nu \otimes \mathcal{K}_\nu \to \mathcal{K}_\nu \otimes \mathcal{K}_\nu \,.$$

Theorem

Consider a principal or complementary series representation, and the integral kernels

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- Form of R essentially fixed by invariance, unitarity, YBE, and crossing.
- Proof of YBE, crossing, ... relies on relations known from analysis of de Sitter Feynman diagrams [Hollands 2012 + Marolf/Morrison 2011], [Hollands 2013]
- Using flat model and principal series reps, YBE was already shown by [Chicherin, Derkachov, Isaev 2001]

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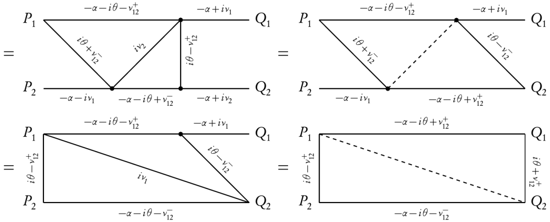
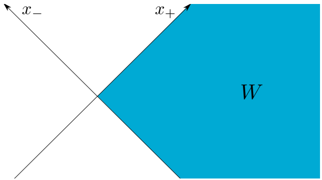


From wedge algebras to full QFTs



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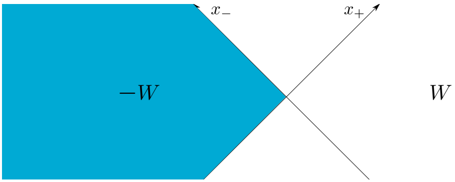

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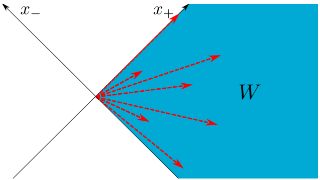

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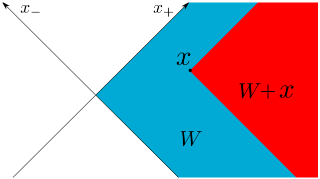

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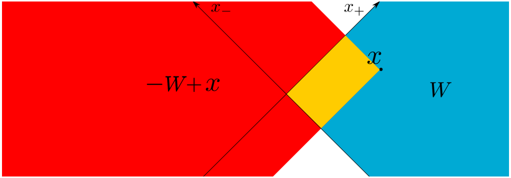

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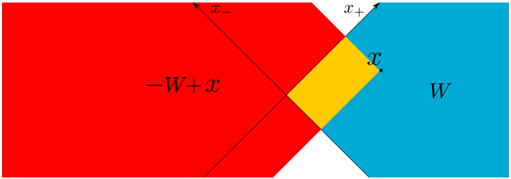

- Any wedge algebra defines a Haag-Kastler net, and any QFT can be obtained from its wedge algebra (in any dimension).
- Elements of **intersections of opposite wedge algebras** are characterized only indirectly. *Existence?*

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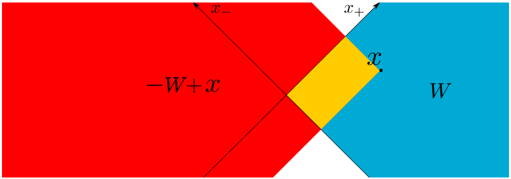

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Theorem: *If* there exist non-trivial local operators, then this construction yields an integrable two-dimensional QFT.

In that case, R represents the two-particle scattering operator of Haag-Ruelle theory, and the theory is even *asymptotically complete*.

Modular nuclearity

A sufficient criterion for the existence of local observables exists:

- **Theorem:** [Buchholz/GL] If the *modular nuclearity condition* of Buchholz-D'Antoni-Longo holds, then "many" local observables exist (cyclic vacuum for double cones).
  This means that

$$\mathcal{M}_R \ni A \longmapsto \Delta^{1/4}_{(\mathcal{M},\Omega)} U(x) A\Omega \,, \qquad x \in W,$$

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- In this case, the inclusions $U(x)\mathcal{M}_R U(x)^{-1} \subset \mathcal{M}_R$, $x \in W$, are split (cf. *Rédei's talk*)

A single particle illustration of modular nuclearity

Consider the Hardy space $H^2 \subset L^2$, and the operator

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which is unbounded.

But if H^2 is completed in the graph norm of $\Delta^{1/2}$ to a Hilbert space (i.e., with scalar product

$$\langle \psi, \varphi \rangle' := \frac{1}{2} \int d\theta \, \left(\overline{\psi(\theta)} \varphi(\theta) + \overline{\psi(\theta + i\pi)} \varphi(\theta + i\pi) \right)$$

), then the operator $\Delta^{1/4} U(x)$ is "almost finite-dimensional" (*s-class*), and in particular nuclear.

Modular Nuclearity in the $O(N)$-model

In the $O(N)$-model, we have a proof of "n-particle nuclearity" based on complex analysis of n-particle wedge-local wavefunctions.

To conclude modular nuclearity / split, we need in addition the so-called "intertwiner property" (an analytic intertwining between two representations of the symmetric/braid group)

Modular Nuclearity in the $O(N)$-model

In the $O(N)$-model, we have a proof of "n-particle nuclearity" based on complex analysis of n-particle wedge-local wavefunctions.

To conclude modular nuclearity / split, we need in addition the so-called "intertwiner property" (an analytic intertwining between two representations of the symmetric/braid group)

Conclusion on $O(d,1)$*-models:*

- The construction of the $O(N)$-sigma models by methods in AQFT is almost complete.
- If the intertwiner property holds, the emerging QFT satisfies the axioms of Haag-Kastler, has the factorizing S-matrix calculated by the Zamolodchikov's, and is asymptotically complete.
- The open intertwiner problem is related to analysis of holomorphic solutions of Yang-Baxter and braid group representations.

A *dS/CFT correspondence* for the $O(d, 1)$ model

For the $O(d, 1)$-invariant R-matrices, we may build from the same data two different models:

- A $O(d, 1)$ sigma model, describing a field on $\mathbb{R}^2$ (or on a lightray) with de Sitter target space $dS_d = SO(d, 1)/SO(d)$
- A CFT on Euclidean $\mathbb{R}^{d-1}$

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- The second version is based on single particle space $\mathbb{C}^N \otimes \mathcal{K}_\nu$, and a choice of N numbers $\theta_1, ..., \theta_N \in \mathbb{R}$. The R-matrix is

$$(R\Psi)_{ij} := R(\theta_i - \theta_j)\Psi_{ji}\,.$$

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- Analogous procedure as before yields R-deformed CCR operators $a_k(\boldsymbol{x})$, $k = 1, ..., N$, such that

$$a_i^\dagger(\boldsymbol{x}_1)a_j(\boldsymbol{x}_2) - \mathcal{R}_{\theta_i-\theta_j}\, a_j(\boldsymbol{x}_2)a_i^\dagger(\boldsymbol{x}_1) = c_\nu \delta_{ij} \cdot |\boldsymbol{x}_1 - \boldsymbol{x}_2|^{-2s}$$
$$a_i^\dagger(\boldsymbol{x}_1)a_j^\dagger(\boldsymbol{x}_2) - \mathcal{R}_{\theta_i-\theta_j}\, a_j^\dagger(\boldsymbol{x}_2)a_i^\dagger(\boldsymbol{x}_1) = 0 \, .$$

with $s = \frac{d-1}{2} - i\nu$.

A *dS/CFT correspondence* for the $O(d,1)$ model

- GNS representation w.r.t. a “vacuum state” yield representation space on which conformal symmetry group of $\mathbb{R}^{d-1}$ acts
- Fields $\phi_j(\boldsymbol{x}) = z_j^\dagger(\boldsymbol{x}) + z_j(\boldsymbol{x})$ are covariant under V_ν, but not “local” (in the sense of permutation symmetric correlation functions) because of the $R(\theta_i - \theta_j)$.

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Conclusion on $O(d,1)$-models:

- $SO(d,1)$-invariant crossing-symmetric Yang-Baxter operators exist and yield different QFT models: $SO(d,1)$-sigma models and Eucl. CFT on $\mathbb{R}^{d-1}$.
- Both cases are generated by non-local fields, but might have also have local fields.
- The two models are related by the same input data (R, V). Currently we do not have a more direct link.
- The CFTs come with a discretization parameter N. Might give rise to a dS/CFT correspondence in the limit $N \to \infty$.