The geometry of regularity structures and "gaugeoid fields"

Mathematics of interacting QFT models, University of York

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Regularity structures: what for and how?

In order to

- control **singularities** (due to the presence of white noise) arising from a fixed point method to solve a semilinear PDE (stochastic PDE) driven by some very singular (typically random) input;
- $\mathcal{L}(u) = F(u, \xi)$, $\mathcal{L}$ (typically parabolic but possibly elliptic) differential operator, ξ is a typically **very irregular random input**,and F is some nonlinearity e.g.
  - KPZ (Kardar-Parisi-Zhang) $\partial_t u = \partial_x^2 u + \underbrace{(\partial_x u)^2 + \xi}_{F(u,\xi)}$,
  - φ^4 model in 3-dim $\partial_t \varphi = \Delta\varphi \underbrace{-\varphi^3 + \xi}_{F(\varphi,\xi)}$;
- in making sense of the resulting **products of distributions**;

By means of an algebraic device which involves

- a graded structure (via the set A) reminiscent of the graded structure arising in Taylor expansions;
- in order to renormalize (the singularities) by means of a (co)algebraic approach governing the addition of diverging counterterms,
- while keeping track of power counting of divergences, a procedure familiar to physicists.

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Our aims

- investigate the **geometry** of regularity structures,
- leaving aside the analytic aspects;
- unravel the role of $\mathbb{R}^d$ as a **translation group** (symmetry) and as a **displacement** (when differentiating).

Our tools

- **direct connections** on groupoids in order to describe **re-expansion maps** (the displacement)– direct connections were introduced by **Teleman** for frame groupoids and arise in synthetic geometry (**Kock**).
- **gaugeoid transformations** on groupoids in order to understand the **role of the symmetry group**,
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The **model** space T encodes the jet or local expansion of a function (or distribution!) at any given point. The group G translates coefficients from a local expansion around a given point into coefficients for an expansion around a different point.

Regularity structure

A triple (A, T, G), where $A \subset \mathbb{R}$ is a discrete set bounded from below,

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A **model** associates to **abstract** elements in T **concrete** functions or distributions on $\mathbb{R}^d$.

Model for a regularity structure (mod. α-Hölder continuity cdn's)

A **model** $(\Pi, \Gamma = \rho \circ \gamma)$ for the regularity structure built from

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A triple (A, T, G), where $A \subset \mathbb{R}$ is a discrete set bounded from below,

- the **model space** $T = \oplus_{\alpha \in A} T_\alpha$ is an A-graded vector space with $T_0 = \mathbb{R}\,\mathbf{1} \simeq \mathbb{R}$ and $\dim(T_\alpha) < \infty$ finite dimensional; T_α comes equipped with a norm $\|\cdot\|_\alpha$.
- the **structure group** G is a Lie group acting on T by an action $\rho : G \times T \to T$ such that $g\,\mathbf{1} = \mathbf{1}$ for any $g \in G$ and $(g - Id)\,(T_\alpha) \subset \oplus_{\beta<\alpha} T_\beta$.

A **model** associates to **abstract** elements in T **concrete** functions or distributions on $\mathbb{R}^d$.

Model for a regularity structure (mod. α-Hölder continuity cdn's)

A **model** $(\Pi, \Gamma = \rho \circ \gamma)$ for the regularity structure built from

- a contn's map $\gamma : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow G$ s.t. $\gamma(x,x) = 1_G$ (and $\gamma(x,y)\,\gamma(y,z) = \gamma(x,z)$);
- continuous linear maps $\Pi_x : T \longrightarrow \mathcal{D}'(\mathbb{R}^d)$ s.t. $\Pi_y = \Pi_x\,\Gamma(x,y)$.

Polynomial regularity structures

Jets and polynomial functions

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- $T = \mathfrak{J}^k(\mathbb{R}^d, \mathbb{R})$ is the space of **jets** on R^d, isomorphic to the space of real abstract polynomials in $X_1, \cdots, X_d$ of total degree $\leq k$;
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Hairer's reconstruction theorem

γ-distributions: a function $\widehat{\varphi} : \mathbb{R}^d \to \oplus_{\alpha<\gamma} T_\alpha$ lies in

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$\mathcal{D}^\gamma$ consists of jets of distributions that locally, around any given point x, look like the model distribution $\Pi_x \widehat{\varphi}(x)$.

Reconstructing the distributions if $\Pi_x \tau$ is continuous for any $\tau \in T$

There is a continuous map $\mathcal{R} : \mathcal{D}^\gamma \to D'(\mathbb{R}^d)$ such that

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Regularity structure on a manifold

A **regularity structure** on a Riemannian manifold (M, g) is a couple $(A, \mathbf{T})$ where $A \subset \mathbb{R}$ is a discrete set bounded from below and

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II. Why groupoids? Re-expansion maps

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In order the make sense of the **re-expansion map** Γ on $M \times M$ and the relations

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provided they hold.

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- **Examples:** $\mathcal{L}(\mathcal{P}(M)) = TM \longrightarrow M$; $\mathcal{L}(\mathrm{Iso}(\mathbf{T})) = \mathrm{Hom}(\mathbf{T}) \longrightarrow M$.

Groupoids

- A **groupoid** $\mathcal{G} \overset{t}{\underset{s}{\rightrightarrows}} M$ is a small category in which every morphism γ_{xy} is invertible; we set $\mathcal{G}_x{}^y := t^{-1}(x) \cap s^{-1}(y)$;
- A groupoid whose structure maps are smooth maps and whose source map s and target map t are surjective submersions, i.e. fibrations, is a **Lie groupoid** ;
- A **trivial Lie** groupoid is of the form $M \times M \times G$, e.g. the pair groupoid $\mathcal{P}(M) = M \times M \rightrightarrows M$ ($G = \{1\}$);
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Direct connections

- Direct connections: **local maps** $\Gamma : \mathcal{P}(M) \ast\!\!\longrightarrow \mathcal{G}$ (i.e. defined on a neighborhood of the diagonal $\Delta \in M \times M$ and $\Gamma(x, y) \in \mathcal{G}_x^y$) such that $\Gamma(x, x) = \mathrm{Id}_x$ (Condition (0));
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Warning: the converse does not hold in general. Not all direct connections come from lifting an infinitesimal connection. Yet they do when they are flat!

Flat direct connections $\longleftrightarrow$ flat infinitesimal connections

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From infinitesimal to direct connections: the general case:

Path connection: parallel transport along paths

infinitesimal connection $\nabla \longleftrightarrow$ path connection $\widetilde{\nabla}$

$$\widetilde{\nabla} : \mathfrak{P}(M) \to \mathfrak{P}^s_M(\mathcal{G}); \quad \frac{d\widetilde{\nabla}(\nu)}{dt}(t_0) = T(R_{\widetilde{\nabla}(\nu(t_0))})\left(\nabla\left(\frac{d\nu}{dt}(t_0)\right)\right).$$

Holonomy map

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A direct connection from an infinitesimal connection ∇

- We lift pairs of points to paths (e.g using geodesics) via a local map $\eta : \mathcal{P}(M) \ni (x, y) \longrightarrow \nu(x, y) \in \mathfrak{P}(M)$
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- We lift pairs of points to paths (e.g using geodesics) via a local map $\eta : \mathcal{P}(M) \ni (x,y) \longrightarrow \nu(x,y) \in \mathfrak{P}(M)$
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From infinitesimal to direct connections: the general case:

Path connection: parallel transport along paths

infinitesimal connection $\nabla \longleftrightarrow$ path connection $\widetilde{\nabla}$

$$\widetilde{\nabla} : \mathfrak{P}(M) \to \mathfrak{P}^s_M(\mathcal{G}); \quad \frac{d\widetilde{\nabla}(\nu)}{dt}(t_0) = T(R_{\widetilde{\nabla}(\nu(t_0))})\left(\nabla\left(\frac{d\nu}{dt}(t_0)\right)\right).$$

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III. Geometric regularity structures

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- $T = \oplus_{\alpha \in A} T_\alpha$ is an A-graded vector space with $T_0 \simeq \mathbb{R}$,
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An A-graded **geometric regularity structure** on a (closed) manifold M is a t-uple $(A, \mathbf{T}, \mathcal{G}(\mathbf{P}), \Gamma)$ built from a regularity structure (A, T, G, ρ) in the following way

1. $\mathbf{P} \to M$ is a G-principal bundle and $\mathcal{G}(\mathbf{P})$ the associated gauge groupoid;
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An A-graded **geometric regularity structure** on a (closed) manifold M is a t-uple $(A, \mathbf{T}, \mathcal{G}(\mathbf{P}), \Gamma)$ built from a regularity structure (A, T, G, ρ) in the following way

1. $\mathbf{P} \to M$ is a G-principal bundle and $\mathcal{G}(\mathbf{P})$ the associated gauge groupoid;
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Geometric Regularity structure

Regularity structure

A triple (A, T, G, ρ), where $A \subset \mathbb{R}$ is a discrete set bounded from below and

- $T = \oplus_{\alpha \in A} T_\alpha$ is an A-graded vector space with $T_0 \simeq \mathbb{R}$,
- G is a Lie group (the symmetry group in physics) acting on T by an action $\rho : G \times T \to T$ such that $(\rho(G) - Id)(T_\alpha) \subset \oplus_{\beta < \alpha} T_\beta$.

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Example: trivial direct connection (rough paths and polynomial reg. str.)

The geometric data

- E, F two vector spaces, $A \subset \mathbb{Z}_{\geq 0}$ and

$$T = \bigoplus_{n \in A} T_n, \quad T_n := \mathcal{L}_s(E^n, F),$$

  where we have set $T_0 = F$.
- $\mathbf{T} = M \times T$ is a trivial bundle;
- the structure group G and the corresponding groupoid $\mathcal{G} = M \times M \times G$;
- an action $\tau : G \times E \longrightarrow E$, and the induced action on T;
- The induced action of $\mathcal{G}$ on $\mathbf{T}$:

$$\begin{array}{rccc} \rho : & \mathcal{G}(\mathbf{P}) & \longrightarrow & \mathrm{Iso}(\mathbf{T}) \\ & (x, y, g) & \longrightarrow & ((y, t) \longmapsto (x, \tau(g)(t))) \end{array}$$

- A map $W : M \to G$ gives rise to a (trivial) direct connection

$$\begin{array}{rcc} \gamma^W : M \times M & \ast\!\!\longrightarrow & G \\ (x, y) & \longmapsto & W(x)W(y)^{-1}; \end{array}$$

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- The representation

$$\begin{array}{rcl} \rho : G & \longrightarrow & \mathrm{Aut}(T) \\ h & \longmapsto & \rho(h) : t \longmapsto t + h\,1; \end{array}$$

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Two examples

- **Rough path** regularity structure: $A = \{0, 1\}$, $M = \mathbb{R}^d$, $G = E$ acting on E by translations, $W : \mathbb{R} \to E$;
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Polynomial regularity structure on a manifold revisited

The geometric data

- (M, g) an n-dimensional Riemannian manifold;
- ∇ the Levi-Civita connection and τ_γ the parallel transport along the curve γ;
- $\mathbf{T} = \mathcal{J}^\bullet(\mathbf{V}) \longrightarrow M$, the (filtered) **jet bundle** of the vector bundle $\mathbf{V} \to M$ with typical fibre V (e.g. $\mathbf{V} = M \times \mathbb{R}$ trivial);
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$$\mathcal{G}\left(\mathrm{GL}(\mathbf{T})\right) = \mathrm{Iso}\left(\mathbf{T}\right) \ni \mathcal{J}^\bullet f_{yx} : \mathcal{J}^\bullet_x(\mathbf{V}) \longrightarrow \mathcal{J}^\bullet_y(\mathbf{V}).$$

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Jet prolongation vs. groupoids

Jet prolongation acts functorially (Kolar and Brno school)

- on fibre bundles $\mathbf{F} \longmapsto \mathcal{J}^\bullet(\mathbf{F})$ and on vector bundles $\mathbf{V} \longmapsto \mathcal{J}^\bullet(\mathbf{V})$;
- but **not on principal bundles**

$$\mathbf{P} \longmapsto \mathcal{W}^\bullet(\mathbf{P}) = \mathbf{F}^\bullet(M) \times \mathcal{J}^\bullet(\mathbf{P}),$$

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- and also on frame groupoids!

$$\mathrm{Iso}(\mathbf{T}) = \mathrm{Iso}(\mathcal{J}^\bullet(\mathbf{V})) = \mathcal{G}(\mathrm{GL}(\mathcal{J}^\bullet(\mathbf{V}))) = \mathcal{J}^\bullet(\mathcal{G}(\mathrm{GL}(\mathbf{V}))) = \mathcal{J}^\bullet(\mathrm{Iso}(\mathbf{V})).$$

Definition

A **geometric polynomial structure** on a vector bundle $\mathbf{V} \to M$ over a Riemannian manifold (M, g) is a quadruple

$$(A, \mathbf{T}, \mathcal{G}(\mathbf{P}), \rho)\,,$$

where $A = \mathbb{Z}_{\geq 0}$, $\mathbf{P} = \mathrm{Iso}(\mathbf{T})$, ρ is the canonical representation, where

- The **model space** is a jet prolonged vector bundle $\mathbf{T} = \mathcal{J}^k(\mathbf{V})$;
- The **re-expansion map** Γ^k_g is the **direct connection** on the **frame groupoid** $\mathrm{Iso}\left(\mathcal{J}^k(\mathbf{V})\right)$ built from the exponential prolongation of a connection $\gamma_{\mathbf{V},g}$ on $\mathbf{V}$;
- The **model** is the map

$$\begin{aligned} \Pi_p : \mathcal{J}^n_p(\mathbf{V}) &\longrightarrow \mathcal{C}(M, \mathbf{V}) \\ \mathrm{j}^n_p(f) &\longmapsto \sum_{k=0}^{n} \partial^k_p(f(\exp^g_p(\bullet))). \end{aligned}$$

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Results

Regularity structures are indeed geometric regularity structures

- We saw that the rough path regularity structure is indeed a **geometric** regularity structure;
- **Theorem 1:** Driver, Dahlqvist and Diehl's polynomial regularity structure is indeed a (local) **geometric polynomial regularity** structure with $\mathbf{V} = M \times \mathbb{R}$;
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V. Gauge theory vs. "Gaugeoid" theory

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Gauge theory from a geometric perspective

Gauge field $\longleftrightarrow$ principal connection

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equivalently,

$$\Phi \in \Omega^1(\mathbf{P}, T\mathbf{P}), \text{ such that } {R_g}^*\Phi = \Phi.$$

In the trivial bundle case $\mathbf{P} = M \times G \Longrightarrow \nabla_{\mathbf{P}} : TM \longrightarrow \mathfrak{g}$.

Gauge transformations $\longleftrightarrow$ Automorphisms of the principal bundle

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Principal bundles with connections vs. gauge groupoids with connections

Gauge groupoids (Ehresman, Mackenzie)

There is a one to one correspondence

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Extension to gauge groupoids with connection

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Flat connections are locally trivial

Flat direct connections $\Gamma(x, y)$ are **trivialisable** i.e., they are **locally** of the form

$$\Gamma^i(x, y) = F^i_x \left(F^i_y\right)^{-1}$$

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"Gaugeoid theory" $\rightsquigarrow$ Geometry of gauge groupoids

- gaugeoid fields are direct connections (or re-expansion maps) on the gauge groupoid $\mathcal{G}(\mathbf{P}) = \mathbf{P} \times_G \mathbf{P} \rightrightarrows M$;
- if $\mathbf{P} = \mathrm{GL}(\mathbf{T})$, gaugeoid fields lie in $\mathrm{Iso}(\mathbf{T}) = \mathcal{G}(\mathrm{GL}(\mathbf{T}))$;
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