

The geometry of regularity structures and "gaugeoid fields"

Mathematics of interacting QFT models, University of York

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Regularity structures: what for and how?

In order to

- ▶ control **singularities** (due to the presence of white noise) arising from a fixed point method to solve a **semilinear PDE** (**stochastic PDE**) driven by some very singular (typically random) input;
- ▶ $\mathcal{L}(u) = F(u, \xi)$, $\mathcal{L}$ (typically parabolic but possibly elliptic) differential operator, ξ is a typically **very irregular random input**, and F is some nonlinearity e.g.
 - ▶ KPZ (Kardar-Parisi-Zhang) $\partial_t u = \partial_x^2 u + \underbrace{(\partial_x u)^2}_{F(u, \xi)} + \xi$,
 - ▶ φ^4 model in 3-dim $\partial_t \varphi = \Delta \varphi - \underbrace{\varphi^3}_{F(\varphi, \xi)} + \xi$;
- ▶ in making sense of the resulting **products of distributions**;

By means of an algebraic device which involves

- ▶ a graded structure (via the set A) reminiscent of the graded structure arising in Taylor expansions;
- ▶ in order to **renormalize** (the singularities) by means of a (co)algebraic approach governing the addition of **diverging counterterms**,
- ▶ while keeping track of **power counting** of divergences, a procedure familiar to physicists.

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Our aims

- ▶ investigate the **geometry** of **regularity structures**,
- ▶ leaving aside the analytic aspects;
- ▶ unravel the role of $\mathbb{R}^d$ as a **translation group** (symmetry) and as a **displacement** (when differentiating).

Our tools

- ▶ **direct connections** on groupoids in order to describe **re-expansion maps** (the displacement)— direct connections were introduced by Teleman for frame groupoids and arise in synthetic geometry (Kock).
- ▶ **gaugeoid transformations** on groupoids in order to understand the **role of the symmetry group**,
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Hairer's setup (2013)

The **model** space T encodes the **jet or local expansion** of a function (or distribution!) at any given point. The group G translates coefficients from a local expansion **around a given point** into coefficients for an expansion **around a different point**.

Regularity structure

A triple (A, T, G) , where $A \subset \mathbb{R}$ is a discrete set bounded from below,

- ▶ the **model space** $T = \bigoplus_{\alpha \in A} T_\alpha$ is an A -graded vector space with $T_0 = \mathbb{R} \mathbf{1} \simeq \mathbb{R}$ and $\dim(T_\alpha) < \infty$ finite dimensional; T_α comes equipped with a norm $\|\cdot\|_\alpha$.
- ▶ the **structure group** G is a Lie group acting on T by an action $\rho : G \times T \rightarrow T$ such that $g \mathbf{1} = \mathbf{1}$ for any $g \in G$ and $(g - \text{Id})(T_\alpha) \subset \bigoplus_{\beta < \alpha} T_\beta$.

A **model** associates to **abstract** elements in T **concrete** functions or distributions on $\mathbb{R}^d$.

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- ▶ a contn's map $\gamma : \mathbb{R}^d \times \mathbb{R}^d \rightarrow G$ s.t. $\gamma(x, x) = 1_G$ (and $\gamma(x, y) \gamma(y, z) = \gamma(x, z)$);
- ▶ continuous linear maps $\Pi_x : T \rightarrow \mathcal{D}'(\mathbb{R}^d)$ s.t. $\Pi_y = \Pi_x \Gamma(x, y)$.

Hairer's setup (2013)

The **model** space T encodes the **jet or local expansion** of a function (or distribution!) at any given point. The group G translates coefficients from a local expansion **around a given point** into coefficients for an expansion **around a different point**.

Regularity structure

A triple (A, T, G) , where $A \subset \mathbb{R}$ is a discrete set bounded from below,

- ▶ the **model space** $T = \bigoplus_{\alpha \in A} T_\alpha$ is an **A-graded vector space** with $T_0 = \mathbb{R} \mathbf{1} \simeq \mathbb{R}$ and $\dim(T_\alpha) < \infty$ finite dimensional; T_α comes equipped with a norm $\|\cdot\|_\alpha$.
- ▶ the **structure group** G is a **Lie group** acting on T by an action $\rho : G \times T \rightarrow T$ such that $g \mathbf{1} = \mathbf{1}$ for any $g \in G$ and $(g - Id)(T_\alpha) \subset \bigoplus_{\beta < \alpha} T_\beta$.

A **model** associates to **abstract** elements in T **concrete** functions or distributions on $\mathbb{R}^d$.

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Polynomial regularity structures

Jets and polynomial functions

- ▶ $A = \mathbb{Z}_{\geq 0}$ is the grading given by the **degree of homogeneous polynomials**;
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Reconstructing Hölder functions:

A function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ is **α -Hölder** continuous (C^α) for $\alpha > 0$

- ▶ **Definition:** if $\forall x \in \mathbb{R}^d, \exists$ a polynomial P_x such that $|\varphi(x+h) - P_x(x+h)| \lesssim |h|^\alpha$ locally uniformly on x and on $|h| \leq 1$.
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A **regularity structure** on a Riemannian manifold (M, g) is a couple $(A, \mathbf{T})$ where $A \subset \mathbb{R}$ is a discrete set bounded from below and

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II. Why groupoids? Re-expansion maps

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The need for a groupoid setup

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In order to make sense of the **re-expansion map** Γ on $M \times M$ and the relations

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provided they hold.

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- ▶ $\Gamma(x, y)$ is a local section of $(T^* \boxtimes T)^{\text{inv}} \rightarrow M \times M$ with $\Gamma(x, x) = I_x$;
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Direct connections on groupoids

Direct connections

- ▶ **Direct connections:** local maps $\Gamma : \mathcal{P}(M) \star \rightarrow \mathcal{G}$ (i.e. defined on a neighborhood of the diagonal $\Delta \in M \times M$ and $\Gamma(x, y) \in \mathcal{G}_x^y$) such that $\Gamma(x, x) = \text{Id}_x$ (Condition (0));
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Warning: the converse does not hold in general. Not all direct connections come from lifting an infinitesimal connection. Yet they do when they are flat!

Flat direct connections $\longleftrightarrow$ flat infinitesimal connections

- ▶ A direct connection Γ is **flat** (Mackenzie's **local morphism** $\Gamma : \mathcal{P}(M) \circ \rightarrow \mathcal{G}$) if it has **trivial curvature** $\Gamma(x, y) \Gamma(y, z) \Gamma(z, x) = \text{Id}_x$ (Condition (1)).
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From infinitesimal to **direct** connections: the general case:

Path connection: parallel transport along paths

infinitesimal connection $\nabla \longleftrightarrow$ path connection $\tilde{\nabla}$

$$\tilde{\nabla} : \mathcal{P}(M) \rightarrow \mathcal{P}_M^s(\mathcal{G}); \quad \frac{d\tilde{\nabla}(\nu)}{dt}(t_0) = T(R_{\tilde{\nabla}(\nu(t_0))}) \left(\nabla \left(\frac{d\nu}{dt}(t_0) \right) \right).$$

Holonomy map

The **holonomy map** of an infinitesimal connection ∇ on $\mathcal{G}$ along a path $\nu \in \mathcal{P}(M)$:

$$H_{\nabla} : \mathcal{P}(M) \longrightarrow \mathcal{G}; \quad \nu \longmapsto H_{\nabla}(\nu) := \tilde{\nabla}(\nu)|_{t=1}.$$

A direct connection from an infinitesimal connection ∇

- ▶ We **lift** pairs of points to paths (e.g using geodesics) via a local map $\eta : \mathcal{P}(M) \ni (x, y) \longrightarrow \nu(x, y) \in \mathcal{P}(M)$
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A **morphism** of Lie algebroids $\phi: \mathcal{L}(\mathcal{G}_1) \rightarrow \mathcal{L}(\mathcal{G}_2)$ integrates to a (locally trivial Lie) groupoid **local morphism** $\Phi: \mathcal{G}_1 \circ \rightarrow \mathcal{G}_2$ uniquely determined modulo germ equivalence.

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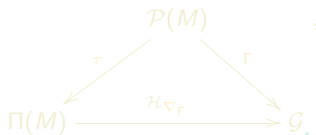
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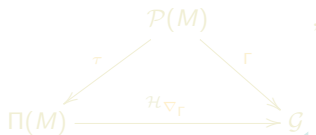
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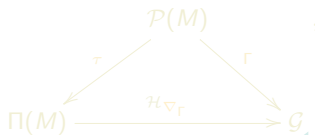
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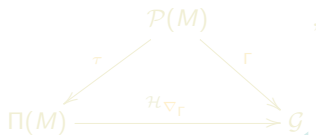
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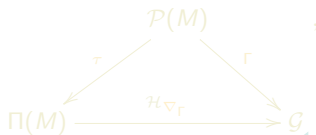
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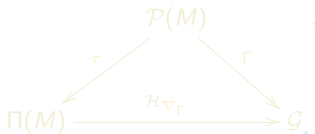
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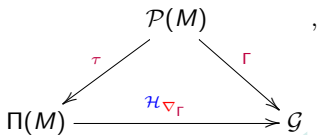
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III. Geometric regularity structures

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Geometric Regularity structure

Regularity structure

A triple (A, T, G, ρ) , where $A \subset \mathbb{R}$ is a discrete set bounded from below and

- ▶ $T = \bigoplus_{\alpha \in A} T_\alpha$ is an A -graded vector space with $T_0 \simeq \mathbb{R}$,
- ▶ G is a Lie group (the symmetry group in physics) acting on T by an action $\rho: G \times T \rightarrow T$ such that $(\rho(G) - \text{Id})(T_\alpha) \subset \bigoplus_{\beta < \alpha} T_\beta$.

A geometric counterpart

An A -graded **geometric regularity structure** on a (closed) manifold M is a t-uple $(A, \mathbf{T}, \mathcal{G}(\mathbf{P}), \Gamma)$ built from a regularity structure (A, T, G, ρ) in the following way

1. $\mathbf{P} \rightarrow M$ is a G -principal bundle and $\mathcal{G}(\mathbf{P})$ the associated gauge groupoid;
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- ▶ $\mathcal{T} = \bigoplus_{\alpha \in A} \mathcal{T}_\alpha$ is an A -graded vector space with $\mathcal{T}_0 \simeq \mathbb{R}$,
- ▶ $\mathcal{G}$ is a Lie group (the symmetry group in physics) acting on $\mathcal{T}$ by an action $\rho: \mathcal{G} \times \mathcal{T} \rightarrow \mathcal{T}$ such that $(\rho(\mathcal{G}) - \text{Id})(\mathcal{T}_\alpha) \subset \bigoplus_{\beta < \alpha} \mathcal{T}_\beta$.

A geometric counterpart

An A -graded geometric regularity structure on a (closed) manifold M is a t-uple $(A, \mathcal{T}, \mathcal{G}(\mathbf{P}), \Gamma)$ built from a regularity structure $(A, \mathcal{T}, \mathcal{G}, \rho)$ in the following way

1. $\mathbf{P} \rightarrow M$ is a G -principal bundle and $\mathcal{G}(\mathbf{P})$ the associated gauge groupoid;
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Example: trivial direct connection (rough paths and polynomial reg. str.)

The geometric data

- ▶ E, F two vector spaces, $A \subset \mathbb{Z}_{\geq 0}$ and

$$T = \bigoplus_{n \in A} T_n, \quad T_n := \mathcal{L}_s(E^n, F),$$

where we have set $T_0 = F$.

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$$\begin{aligned} \rho : \quad \mathcal{G}(\mathbf{P}) &\longrightarrow \text{Iso}(\mathbf{T}) \\ (x, y, g) &\longrightarrow ((y, t) \mapsto (x, \tau(g)(t))) \end{aligned}$$

- ▶ A map $W : M \rightarrow G$ gives rise to a (trivial) direct connection

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- ▶ The representation

$$\begin{aligned}\rho : G &\longrightarrow \text{Aut}(\mathcal{T}) \\ h &\longmapsto \rho(h) : t \longmapsto t + h1;\end{aligned}$$

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- ▶ **Rough path** regularity structure: $A = \{0, 1\}$, $M = \mathbb{R}^d$, $G = E$ acting on E by translations, $W : \mathbb{R} \rightarrow E$;
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IV. Polynomial regularity structures

by means of jets prolongations (Kolar, Michor, Slovák)

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The geometric data

- ▶ (M, g) an n -dimensional Riemannian manifold;
- ▶ ∇ the Levi-Civita connection and τ_γ the parallel transport along the curve γ ;
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- ▶ $\mathbf{T} = \mathcal{J}^\bullet(\mathbf{V}) \rightarrow M$, the (filtered) **jet bundle** of the vector bundle $\mathbf{V} \rightarrow M$ with typical fibre V (e.g. $\mathbf{V} = M \times \mathbb{R}$ trivial);
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Polynomial regularity structure on a manifold revisited

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Jet prolongation vs. groupoids

Jet prolongation acts functorially (Kolar and Brno school)

- ▶ on fibre bundles $\mathbf{F} \mapsto \mathcal{J}^\bullet(\mathbf{F})$ and on vector bundles $\mathbf{V} \mapsto \mathcal{J}^\bullet(\mathbf{V})$;
- ▶ but **not on principal bundles**

$$\mathbf{P} \mapsto \mathcal{W}^\bullet(\mathbf{P}) = \mathbf{F}^\bullet(M) \times \mathcal{J}^\bullet(\mathbf{P}),$$

$\mathbf{F}^k(M)$ k -th order frame bundle consisting of all k -th jets with source 0 of the local diffeomorphisms of $f : \mathbb{R}^n \rightarrow M$;

- ▶ act functorially on **groupoids**!

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- ▶ and also on **frame groupoids**!

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Geometric polynomial regularity structure

Definition

A **geometric polynomial structure** on a vector bundle $\mathbf{V} \rightarrow M$ over a Riemannian manifold (M, g) is a quadruple

$$(A, \mathbf{T}, \mathcal{G}(\mathbf{P}), \rho),$$

where $A = \mathbb{Z}_{\geq 0}$, $\mathbf{P} = \text{Iso}(\mathbf{T})$, ρ is the canonical representation, where

- ▶ The **model space** is a jet prolonged vector bundle $\mathbf{T} = \mathcal{J}^k(\mathbf{V})$;
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Regularity structures are indeed geometric regularity structures

- ▶ We saw that the rough path regularity structure is indeed a geometric regularity structure;
- ▶ **Theorem 1:** Driver, Dahlqvist and Diehl's polynomial regularity structure is indeed a (local) geometric polynomial regularity structure with $\mathbf{V} = M \times \mathbb{R}$;
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equivalently,

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Principal bundles with connections vs. gauge groupoids with connections

Gauge groupoids (Ehresman, Mackenzie)

There is a one to one correspondence

$$\text{principal bundles } \mathbf{P} \longleftrightarrow \text{gauge groupoids } \mathcal{G}(\mathbf{P})$$

Extension to gauge groupoids with connection

Theorem 3: There is a one to one correspondence

$$\text{flat principal bundles } (\mathbf{P}, \nabla) \longleftrightarrow \text{flat gauge groupoids } (\mathcal{G}(\mathbf{P}), \Gamma_{\nabla}).$$

Flat connections are locally trivial

Flat direct connections $\Gamma(x, y)$ are **trivialisable** i.e., they are **locally** of the form

$$\Gamma^i(x, y) = F_x^i \left(F_y^i \right)^{-1}$$

with the F^i 's given in terms of local sections of $\mathbf{P}$ (local section atlas of the groupoid). 

Principal bundles with connections vs. gauge groupoids with connections

Gauge groupoids (Ehresman, Mackenzie)

There is a one to one correspondence

$$\text{principal bundles } \mathbf{P} \longleftrightarrow \text{gauge groupoids } \mathcal{G}(\mathbf{P})$$

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- ▶ $\text{Aut}(\mathcal{G}) := \{\Phi \in \text{Diff}(\mathcal{G}), \Phi(\gamma_1 \gamma_2) = \Phi(\gamma_1) \Phi(\gamma_2) \text{ for composable } (\gamma_1, \gamma_2) \in \mathcal{G}^2\} \ni \text{diffeomorphisms of } \mathcal{G} \text{ which are groupoid morphisms,}$
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$$Z_G^\infty(\mathbf{P} \times \mathbf{P}, \text{Ad}(\mathbf{G}))/B_G^\infty(\mathbf{P} \times \mathbf{P}, \text{Ad}(\mathbf{G})) \simeq \text{Aut}_M(\mathcal{G}(\mathbf{P}))/\text{Aut}_M(\mathbf{P}) \text{Aut}_M(\mathbf{P})^{-1},$$

$$\text{with } \text{Aut}_M(\mathbf{P}) \text{Aut}_M(\mathbf{P})^{-1}$$

$$:= \{\Gamma \in \text{Aut}_M(\mathcal{G}(\mathbf{P})), \Gamma(p, q) = F(p) F(q)^{-1}, \text{ for an } F \in \text{Aut}_M(\mathbf{P})\}.$$



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- ▶ gaugeoid fields are direct connections (or re-expansion maps) on the gauge groupoid $\mathcal{G}(\mathbf{P}) = \mathbf{P} \times_{\mathcal{G}} \mathbf{P} \rightrightarrows M$;
- ▶ if $\mathbf{P} = \mathrm{GL}(\mathbf{T})$, gaugeoid fields lie in $\mathrm{Iso}(\mathbf{T}) = \mathcal{G}(\mathrm{GL}(\mathbf{T}))$;
- ▶ gaugeoid transformations $\in \mathrm{Aut}_M(\mathcal{G}(\mathbf{P}))$ act on direct connections (or re-expansion maps) by composition;
- ▶ the field strength of the "gaugeoid" field is the curvature of the direct connection, so the **obstruction** to the flatness of Γ (Condition (1)).

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