The propagator of spectral QED

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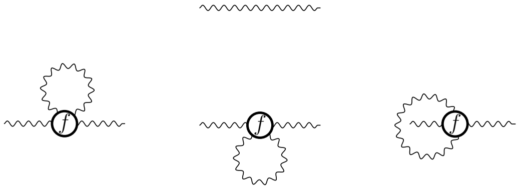


Part 1:

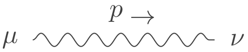
Motivation

The Maxwell action

For $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the Maxwell action equals

$$\begin{aligned} S[A] &= \tfrac{1}{4} \int_{\mathbb{R}^d} \mathrm{d}x \, F_{\mu\nu}(x) F^{\mu\nu}(x) \\ &= \tfrac{1}{4} \int_{\mathbb{R}^d} \mathrm{d}x \, (\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x))(\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)) \\ &= \tfrac{1}{2} \int_{\mathbb{R}^d} \frac{\mathrm{d}p}{(2\pi)^d} \, \hat{A}_\mu(p) \hat{A}_\nu(-p)(-g^{\mu\nu} p^2 + p^\mu p^\nu). \end{aligned}$$

This is put in $\int \mathrm{d}[A] e^{i\hbar^{-1} S[A]}$ and out comes the propagator



$$= \Big(g^{\mu\nu} + (\xi - 1) \frac{p^\mu p^\nu}{p^2} \Big),$$

for gauge fixing parameter ξ. Together with the other Feynman rules of QED, this leads to experimental predictions.

Hold on. Could

$$S[A] = \int_{\mathbb{R}^d} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

be the remnant of something operator algebraic? We vaguely recognize the operators

$$A = \gamma^\mu A_\mu, \qquad D = -i\gamma^\mu \partial_\mu$$

that also appear in $\overline{\psi}(D + A)\psi$, e.g., $[D, A] = -i\gamma^\mu\gamma^\nu\partial_\mu A_\nu$.

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Surely, it won't depend on the way $D + A$ is represented on the Hilbert space. So Connes and Chamseddine conjectured:

> **The action depends only on the class of $D + A$ modulo unitary invariance (i.e., on its spectrum).**

If the action behaves nicely under direct sums, it follows that

$$S[A] = \mathrm{Tr}(f(D + A)) \qquad (\textbf{the spectral action})$$

for some function $f : \mathbb{R} \to \mathbb{R}$. Sanity check:
$\mathrm{Tr}(f(U(D + A)U^*)) = \mathrm{Tr}(Uf(D + A)U^*) = \mathrm{Tr}(f(D + A))$.
Remarkably, in the weak-field limit, (truncated asymptotic expansion of $\mathrm{Tr}(f(\frac{D+A}{\Lambda}))$ as $\Lambda \to \infty$), one exactly recovers the Maxwell action.

Gauge invariance

Writing

$$\begin{aligned} D &\mapsto D^U = D \\ A &\mapsto A^U = UAU^* + U[D, U^*] \end{aligned}$$

one sees that $D^U + A^U = D + U[D, U^*] + UAU^* = U(D + A)U^*$.
A subclass of unitaries is given by

$$U\psi(x) = e^{i\alpha(x)}\psi(x).$$

One obtains

$$A^U_\mu = (e^{i\alpha} A e^{-i\alpha} + e^{i\alpha} i\gamma^\nu \partial_\nu \alpha e^{-i\alpha})_\mu = A_\mu + i\partial_\mu \alpha,$$

i.e., the spectral action is gauge invariant. But much more is true.

- The real reason for believing in the spectral action is that it recovers GR+the full standard model, including e.g. the Higgs as a gauge field and $U(1) \times SU(2) \times SU(3)$, from simple axioms that encode geometry in operator algebra (Chamseddine, Connes, Mukhanov, 2014).
- It uses the spectral action, where instead of $(C_0(\mathbb{R}^d), L^2(\mathbb{R}^d) \otimes \mathbb{C}^s, D = -i\gamma^\mu \partial_\mu)$ more general spectral triples $(\mathcal{A}, \mathcal{H}, D)$ are considered.
- Examples are noncommutative planes, which led to great progress towards construction of NCϕ^4 (Grosse, Wulkenhaar, Rivasseau, ...). The spectral action is the obvious next step.
- Quantization of the spectral action is hard.
- A spectral action matrix-model was introduced in [vN& van Suijlekom, 2022 & 2022], leading to one-loop Gomis–Weinberg renormalizability. But the role of $\mathcal{A}$ was unclear.
- So let us go back to Maxwell, which should emerge from $\mathcal{A} = \mathcal{S}(\mathbb{R}^d) \subseteq C_0(\mathbb{R}^d)$.

Part 2:

Towards quantization of the spectral action

Step 1: subtract a constant term

The Feynman path integral (Wick-rotated to Euclidean signature) is

$$\frac{\int \mathrm{d}[A] e^{-\hbar^{-1}\,\mathrm{Tr}(f(D+A))} G[A]}{\int \mathrm{d}[A] e^{-\hbar^{-1}\,\mathrm{Tr}(f(D+A))}} = \frac{\int \mathrm{d}[A] e^{-\hbar^{-1}\,\mathrm{Tr}(f(D+A)-f(D))} G[A]}{\int \mathrm{d}[A] e^{-\hbar^{-1}\,\mathrm{Tr}(f(D+A)-f(D))}},$$

so we may as well consider $S[A] = \mathrm{Tr}(f(D+A) - f(D))$, thus answering the question from the audience.

Step 2: Taylor expand

How should you compute

$$\frac{\int \mathrm{d}[A] e^{-\hbar^{-1}\,\mathrm{Tr}(f(D+A)-f(D))} G[A]}{\int \mathrm{d}[A] e^{-\hbar^{-1}\,\mathrm{Tr}(f(D+A)-f(D))}} = ?$$

Use the noncommutative Taylor expansion:

$$\begin{aligned}
&\mathrm{Tr}(f(D+A)-f(D)) \\
&= \sum_{n=1}^{\infty} \frac{1}{n!}\,\mathrm{Tr}\left(\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(D+tA)\Big|_{t=0}\right) \\
&= \sum_{n=1}^{\infty} \mathrm{Tr}\left(\int_{\mathbb{R}} \mathrm{d}t\, \widehat{f^{(n)}}(t) \int_{\Delta_n} \mathrm{d}s\, e^{its_0 D} A e^{its_1 D} \cdots A e^{its_n D}\right).
\end{aligned}$$

E.g., the first term is $\mathrm{Tr}(f'(D)A)$, linear in A (a tadpole), the second term is quadratic in A, et cetera. All terms are still nonlocal.

Step 3: Tricks of Feynman and Newton.

Feynman wrote numbers on top of operators to encode their ordering:

$$\overset{1}{A}\overset{3}{B}\overset{2}{C} := ACB.$$

In terms of this notation, abbreviate (Krein, Birman, Solomyak),

$$T_\phi^{A_0,\ldots,A_n}(B_1,\ldots,B_n) := \phi(\overset{1}{A_0},\overset{3}{A_1},\overset{5}{A_2},\ldots,\overset{2n+1}{A_n})\overset{2}{B_1}\overset{4}{B_2}\cdots\overset{2n}{B_n}.$$

Next, the Newton divided differences of f' are

$$f'^{[n-1]}(x_1,\ldots,x_n) := \sum_j \frac{f'(x_j)}{\prod_{i\neq j}(x_j - x_i)}.$$

Combining Newton and Feynman,

$$\begin{aligned}&\mathrm{Tr}\Big(\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(D+tA)\Big|_{t=0}\Big)\\&= \int_{\mathbb{R}^{dn}} \mathrm{d}\mathbf{p}\hat{A}_{\mu_1}(p_1)\cdots\hat{A}_{\mu_n}(p_n)\int_{\mathbb{R}^d}\mathrm{d}k\,\mathrm{tr}(\gamma^{\mu_1}T_{f'^{[n-1]}}^{\not p_1+\not k,\not p_1+\not p_2+\not k,\ldots,\not k}(\gamma^{\mu_2},\ldots,\gamma^{\mu_n})).\end{aligned}$$

Result 1: Vanishing tadpole

With these techniques, one easily proves that

Theorem

If $f \in \mathcal{S}(\mathbb{R})$ is even, then for all odd $n \in \mathbb{N}$,

$$\mathrm{Tr}\,\Big(\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(D+tA)\big|_{t=0}\Big) = 0.$$

In particular for $n=1$: the tadpole term vanishes.

This extends [Iochum/Levy(2011)].

So, $S[A] = \mathrm{Tr}(\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(D+tA)\big|_{t=0}) + \mathrm{Tr}(\frac{\mathrm{d}^4}{\mathrm{d}t^4} f(D+tA)\big|_{t=0}) + \ldots$.

Result 2: The photon propagator of spectral QED

Theorem

Suppose $f(x) = e^{-x^2}$, *then*

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\mathrm{Tr}(f(D+tA))\Big|_{t=0}\\&=\int_{\mathbb{R}^d}\frac{\mathrm{d}p}{(2\pi)^d}\hat{A}_\mu(p)\hat{A}_\nu(-p)\eta_f(p)\Big(\delta^{\mu\nu}p^2-p^\mu p^\nu\Big),\end{aligned}$$

where η_f *is a radial function whose only zero is at* $p = 0$*(!!!), namely*

$$\eta_f(p) = \frac{4}{2^d\pi^{d/2}}\frac{1}{p^2}\Big(\varphi(p) - 1 + \tfrac{1}{2}p^2\varphi(p)\Big),$$

in terms of

$$\varphi(p) := \frac{\sqrt{\pi}e^{-p^2}\,\mathrm{erfi}(\|p\|)}{2\|p\|}.$$

This extends to more general f by Laplace transform, and recovers some asymptotic formulas by van Suijlekom.

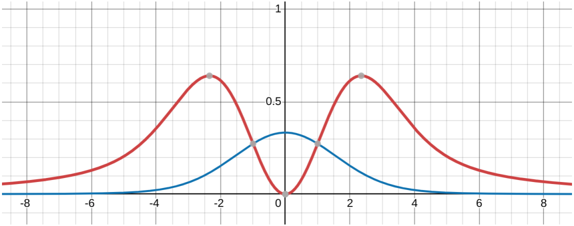
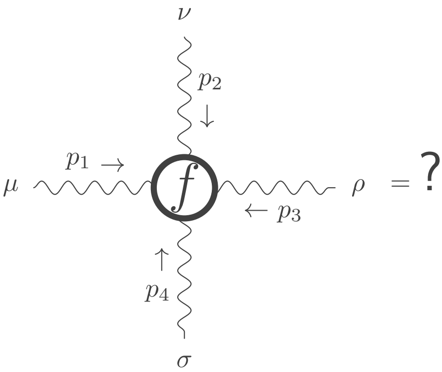


Figure: The blue curve is η_f. The blue curve is $\varphi(k) - 1 + \frac{1}{2}k^2\varphi(k)$. (Of course the functions depicted are slices of radial functions.)

$$\mu \;\overset{p\,\rightarrow}{\sim\!\sim\!\sim\!\sim\!\sim\!\sim}\; \nu \quad = \frac{1}{p^2\eta_f(p)}\left(\delta^{\mu\nu} + (\xi\eta_f(p) - 1)\frac{p^\mu p^\nu}{p^2}\right),$$

As $p \to 0$, $\eta_f(p) \to 1$, recovering the QED photon propagator. As $p \to \infty$, $\eta_f(p) \to 0$ polynomially, recovering [Iochum, Levy, Vassilevich, 2012].

The next challenge is therefore to **explicitly** compute the fourth order derivative.



It seems that it is a linear combination of $\varphi(p_1), \ldots, \varphi(p_4), \varphi(p_1 + p_2), \varphi(p_1 + p_3)$, whose coefficients are rational functions in $p_1, \ldots, p_4$, with a combinatorial interpretation.

Mathematical question

Let $\Delta_2 := \{(s_1, s_2, s_3) \in [0,1]^3 \ : \ s_1 + s_2 + s_3 = 1\}$.
Let $\mathbb{E}_{\mathbf{s}}(\mathbf{k}) := \sum_{i=1}^{3} s_i k_i$, and let $p_i := k_i - k_{i-1}$ (so that $p_1 + p_2 + p_3 = 0$).

Lemma

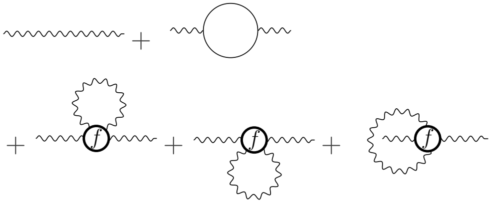
For all $k_1, k_2, k_3 \in \mathbb{R}$ we have

$$\int_{\Delta_2} e^{\mathbb{E}_{\mathbf{s}}(\mathbf{k})^2 - \mathbb{E}_{\mathbf{s}}(\mathbf{k}^2)} \mathrm{d}\mathbf{s} = \frac{\varphi(p_1)}{p_2 p_3} + \frac{\varphi(p_2)}{p_3 p_1} + \frac{\varphi(p_3)}{p_1 p_2}.$$

The left-hand side makes sense for $k_i \in \mathbb{R}^d$, what is the right-hand side?

Outlook

If we have the quartic vertex, we can already calculate the 1-loop quantum corrections to the spectral propagator:



- Ambitious, but in principle possible: calculate the magnetic moment of the electron (constraining f: relevant beyond QED)
- Find a Ward–Takahashi identity
- Power counting. Joint work with Reimann and Hekkelman: spectral action matrix model admits power counting.
- For spectral QED, again divided differences show up, but now more convoluted ones. Free mathematical puzzles! :D

Thanks for listening!