AN INTRODUCTION TO THE YANG-BAXTER EQUATION

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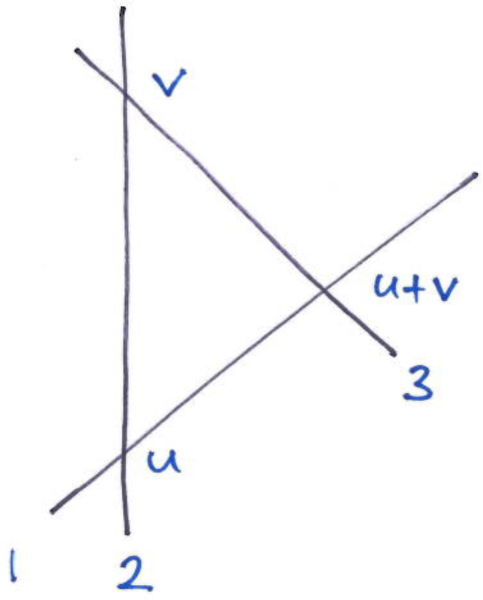
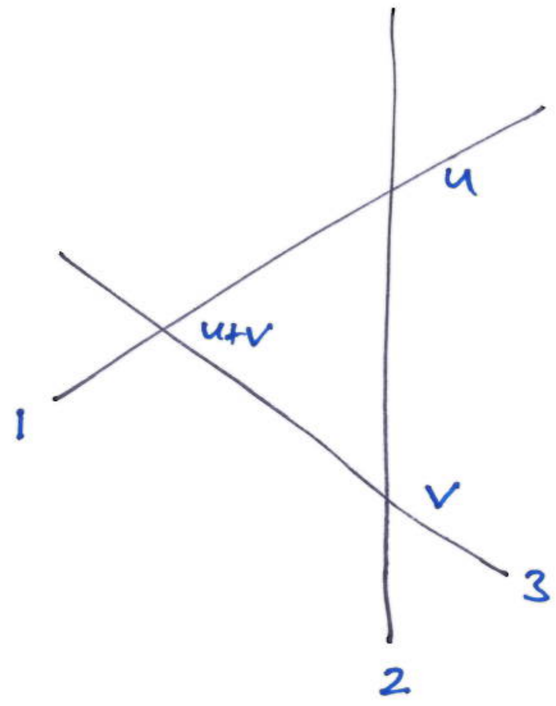
YORK

CARDIFF, 2ND DEC '16

$$R_{12}(u)\, R_{13}(u+v)\, R_{23}(v) = R_{23}(v)\, R_{13}(u+v)\, R_{12}(u)$$





ON

$V_1 \otimes V_2 \otimes V_3$

SOLUTIONS: $V = \mathbb{C}^2$

RATIONAL

$$\begin{pmatrix} 1+u & & & \\ & u & 1 & \\ & 1 & u & \\ & & & 1+u \end{pmatrix}$$

YANG '67

TRIGONOMETRIC

$$\begin{pmatrix} \sin(\eta+u) & & & \\ & \sin u & \sin\eta & \\ & \sin\eta & \sin u & \\ & & & \sin(\eta+u) \end{pmatrix}$$

ELLIPTIC

$$\begin{pmatrix} a & & & d \\ & b & c & \\ & c & b & \\ d & & & a \end{pmatrix}$$

BAXTER '72

LET $u, v \to \infty$.

LHS of YBE is

$$\left(1 + \frac{r_{12}}{u} + O\left(1/u^2\right)\right)\left(1 + \frac{r_{13}}{u+v} + O\left(1/(u+v)^2\right)\right)\left(1 + \frac{r_{23}}{v} + O\left(1/v^2\right)\right)$$

$$\overset{\text{SIMPLEST}}{=} \left(1 + \frac{\sum_a t_a \otimes t_a \otimes 1}{u} + \dots\right)\left(1 + \frac{\sum_b t_b \otimes 1 \otimes t_b}{u+v} + \dots\right)\left(1 + \frac{\sum_c 1 \otimes t_c \otimes t_c}{v} + \dots\right)$$

BELAVIN-DRINFELD

WHERE $\{t_a\}$ GENERATE g (COMPACT),

SINCE LHS - RHS =

$$\frac{1}{uv(u+v)}\left((u+v)\, f_{acb}\, t_a \otimes t_b \otimes t_c + v\, f_{abc}\, t_c \otimes t_a \otimes t_b + u\, f_{bca}\, t_b \otimes t_c \otimes t_a\right) = 0.$$

SO THE YBE $\Rightarrow$ LIE ALGEBRAS !

NOW KEEP u FINITE, LET $v \to \infty$.

$$R_{12}(u)\Big(1 + \underbrace{\frac{1}{u+v}}_{=\,\frac{1}{v} - \frac{u}{v^2} + \cdots} t_c \otimes 1 \otimes t_c + \cdots\Big)\Big(1 + \frac{1}{v}\, 1 \otimes t_b \otimes t_b + \cdots\Big) = \text{RHS}$$

THEN

v^0 : TRIVIAL

v^{-1} : $\left[R(u),\ t_a \otimes 1 + 1 \otimes t_a\right] = 0 \quad \Rightarrow R(u) = \sum_{U \subset V \otimes V} \tau_U(u) P_U$

v^{-2} : $R(u)\left(-u\, t_a \otimes 1 + \frac{1}{2} f_{abc}\, t_b \otimes t_c\right)$

$$= \left(-u\, t_a \otimes 1 - \frac{1}{2} f_{abc}\, t_b \otimes t_c\right) R(u)$$

NOW WRITE

$f_{abc}\, t_b \otimes t_c = \frac{1}{2}\left[C,\ t_a \otimes 1\right]$

$C = (t_b \otimes 1 + 1 \otimes t_b)(t_b \otimes 1 + 1 \otimes t_b)$

$$\Rightarrow \frac{\tau_1}{\tau_2} = \frac{u + \frac{1}{4}(C_1 - C_2)}{u - \frac{1}{4}(C_1 - C_2)}$$

THE YANGIAN : DRINFELD '86

$$\mathfrak{g}: \quad [I_a, I_b] = f_{abc} I_c \qquad \Delta(I_a) = I_a \otimes 1 + 1 \otimes I_a$$

$$\{J_a\}: \quad [I_a, J_b] = f_{abc} J_c \qquad \Delta(J_a) = J_a \otimes 1 + 1 \otimes J_a + \frac{1}{2} f_{abc} I_b \otimes I_c$$

$T_u : I_a \mapsto I_a, \quad J_a \mapsto J_a + u I_a$ IS AN AUTOMORPHISM

$$\rho(I_a) = t_a, \quad \rho(J_a) = 0$$

AND THE YBE IS $T_u \otimes T_0$ APPLIED TO

$$R(u)\, \Delta(x) = \Delta'(x)\, R(u)$$

... AND

$$[J_a, [J_b, I_c]] - [I_a, [J_b, J_c]] = \frac{1}{24} f_{api} f_{bqj} f_{crk} f_{ijk} I_p I_q I_r$$

CLASSICAL INVERSE SCATTERING METHOD: THE <u>LAX PAIR</u>:

RECAST A PDE AS $\left[\partial_x + L_x, \partial_t + L_t\right] = 0$ FOR MATRICES L_x, L_t.

EXAMPLES

$\partial_0 L_1 - \partial_1 L_0 + [L_0, L_1] =$

1/ **PCM** $\partial_\mu j^\mu = 0$

$\partial_0 j_1 - \partial_1 j_0 + [j_0, j_1] = 0$

FOR $j^\mu = j^\mu_a t_a \in \mathfrak{g}$

$$L_0 = \frac{1}{1-u^2}(j_0 - u j_1)$$

$$L_1 = \frac{1}{1-u^2}(j_1 - u j_0)$$

$$(1-u^2)(\partial_0 j_1 - u\partial_0 j_0 - \partial_1 j_0 + u\partial_1 j_1) + [j_0 - u j_1, j_1 - u j_0]$$

2/ **SINE-GORDON** $[\sigma_i, \sigma_j] = i\epsilon_{ijk}\sigma_k$

$$\Box\varphi + \frac{m^2}{\beta}\sin\beta\varphi = 0$$

$$L_0 = i\frac{\beta}{2}(\partial_1\varphi)\sigma_3 + m\sin\left(\frac{\beta\varphi}{2}\right)\sigma_2$$

$$L_1 = i\frac{\beta}{2}(\partial_0\varphi)\sigma_3 - m\cos\left(\frac{\beta\varphi}{2}\right)\sigma_1$$

$$\frac{i\beta}{2}(\partial_0^2 - \partial_1^2)\varphi\,\sigma_3 - m^2\sin\frac{\beta\varphi}{2}\cos\frac{\beta\varphi}{2}[\sigma_2, \sigma_1]$$

+ X-TERMS CANCEL.

$$(\partial_1 + L_1)T = 0$$

$$\Rightarrow T = \mathbb{P}e^{-\int L_1}$$

$$\Rightarrow \frac{d}{dt}T = 0$$

CONSERVED QUANTITIES

AND $\left\{T(u) \underset{,}{\otimes} T(u')\right\} = \left[r(u,u'), T(u)\otimes T(u')\right]$

$\mathrm{Tr} \Rightarrow$ INVOLUTION

THE QUANTUM LAX PROBLEM:

$$T = 1 - \int L_1 + \int L_1 \left(\int^x L_1 \, dy \right) dx + \cdots$$

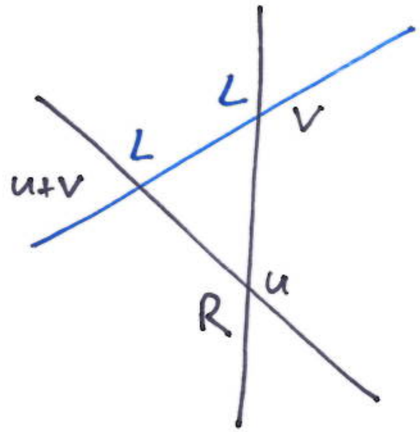
IS PLAGUED BY DIVERGENCES.

DEFINE A LATTICE LAX OPERATOR $L_N = \mathbb{P}\exp \int_{(n-1/2)\Delta}^{(n+1/2)\Delta} L(x,u)\, dx$

AND THESE ARE BROUGHT UNDER CONTROL.

FURTHER $\quad R(u)\, L_N^1(u+v)\, L_N^2(v) = L_N^2(v)\, L_N^1(u+v)\, R(u)$

WHERE $\quad L_N^1 = L_N \otimes 1 \,, \quad L_N^2 = 1 \otimes L_N \,.$



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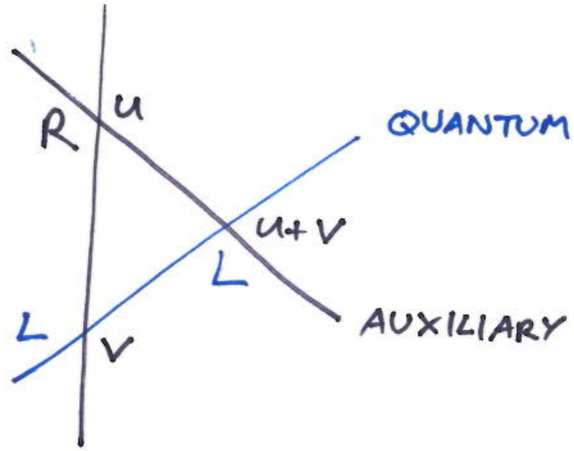


EXAMPLES

1/ PCM : RECALL $j^\mu = \underset{\text{QUANTUM}}{j^\mu_a} \, \underset{\text{AUXILIARY}}{t_a}$

CCR $\quad [j_{N\pm}, j_{M\pm}] = -i\hbar g \, \delta_{NM} \, j_{N\pm} - k \, C_2$

$$\Rightarrow \quad L_N(u) = 1 + \sum_{n=0}^{\infty} \left(\left(\frac{1}{1-u}\right)^{n+1} j^a_{N-} + \left(\frac{-1}{1+u}\right)^{n+1} j^a_{N+} \right) (-1)^n t^a_n$$

$$t^a_n = \left(\frac{i\hbar g}{2}\right)^n t^a$$

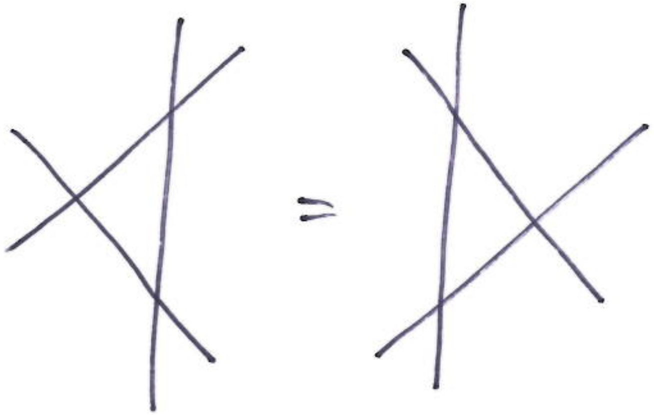
$L^1_N L^2_N$ A GOOD COPRODUCT $\Rightarrow t^a_n$ NOT A LOOP ALGEBRA BUT A YANGIAN

2/ SINE-GORDON : USE $p_N = \underset{\text{ONE STEP}}{\int \dot\varphi}$, $\quad q_N = \int e^{\beta\varphi}$, $\quad [p_N, q_M] = \frac{i\hbar\beta}{2} q_N \, \delta_{NM}$

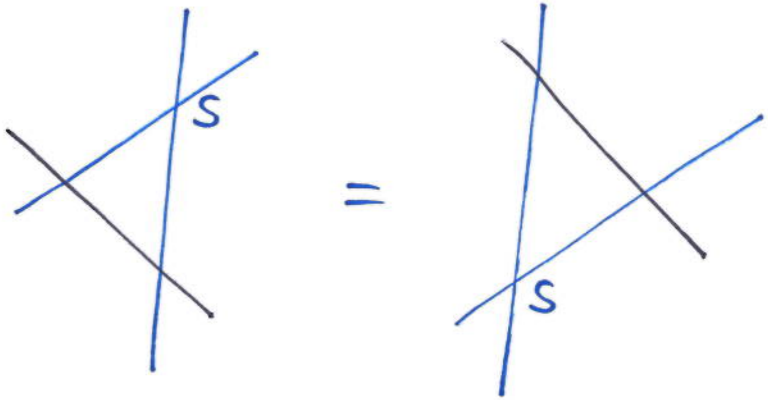
$$\Rightarrow \quad L_N = e^{\frac{\beta H p_N}{2}} + e^{\frac{\beta H p_N}{4}} \frac{m}{2} \left[q_N (E^+ + E^-) + q_N^{-1} \left(\lambda E_0^+ + \frac{1}{\lambda} E_0^- \right) \right] e^{\frac{\beta H p_N}{4}}$$

$L^1_N L^2_N$ A GOOD COPRODUCT $\Rightarrow H, E^\pm, E_0^\pm$ A QUANTIZED AFFINE ALGEBRA

KULISH & RESHETIKHIN (1982) · DRINFELD, JIMBO (1986)

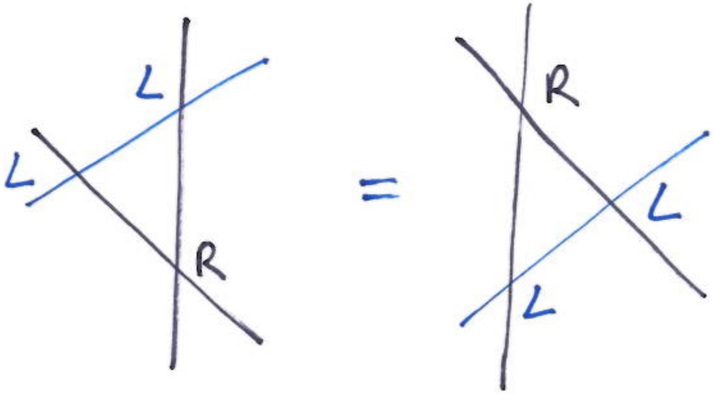


R R R = R R R

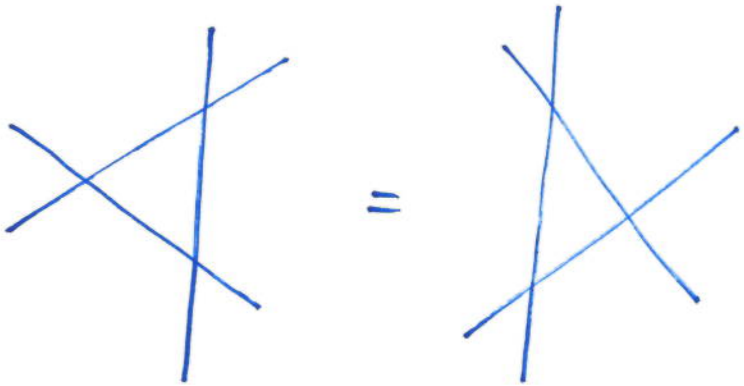


S L L = L L S

CHARGES ARE CONSERVED



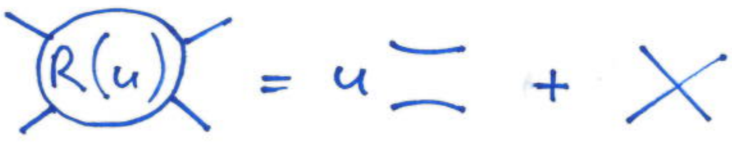
L L R = R L L



S S S = S S S

SCATTERING FACTORIZES

$$\begin{array}{cccc} 11 & 12 & 21 & 22 \\ \end{array}$$
$$\begin{pmatrix} 1+u & & & \\ & u & 1 & \\ & 1 & u & \\ & & & 1+u \end{pmatrix} = uI + P$$



where $\delta_{ij} = i \longrightarrow j$.

THEN $R(u) = u\,\mathbb{1} + \times = \frac{1}{2}(u+1)(\mathbb{1} + \times) + \frac{1}{2}(u-1)(\mathbb{1} - \times)$

$$= (u+1)\left\{ P_+ + \frac{u-1}{u+1} P_- \right\}$$

WHERE $P_\pm = \frac{1}{2}(\mathbb{1} \pm \times)$; $P_\pm^2 = \frac{1}{4}(\mathbb{1}\mathbb{1} \pm \mathbb{1}\times \pm \times\mathbb{1} + \times\times)$

$$= P_\pm$$

ARE SYMMETRIZER & ANTI-SYMMETRIZER AND PROJECT ONTO $\square\square$, $\begin{smallmatrix}\square\\\square\end{smallmatrix}$:
ON $V^{\otimes N'}$

$$\mathbb{C}S_{N'} \xleftrightarrow{\text{SCHUR-WEYL DUALITY}} sl_n$$

(HERE $N'=2$)

$\underline{SO_{2n}}$: $R = u \, = \, + \times + \frac{2n-2}{2n} \frac{u}{u-n+1} \supset\subset$

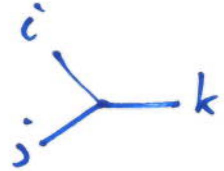
$$= (u+1) P_+ + (u-1) P_- + (u-1) \frac{u+n-1}{u-n+1} P_1$$

WHERE $P_+ = \frac{1}{2}\left(= + \times \right) - \frac{1}{N} \supset\subset \quad P_- = \frac{1}{2}\left(= - \times \right) \quad P_1 = \frac{1}{2n} \supset\subset$

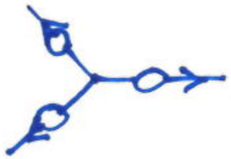
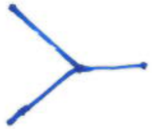
$P_1^2 = \frac{1}{(2n)^2} \supset \bigcirc \subset = \frac{1}{2n} \supset\subset$ SINCE $\delta_{ij}\delta_{ji} = 2n$.

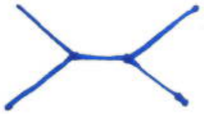
$\langle =, \times, \supset\subset \rangle$ IS BRAUER'S ALGEBRA ;

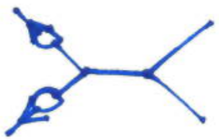
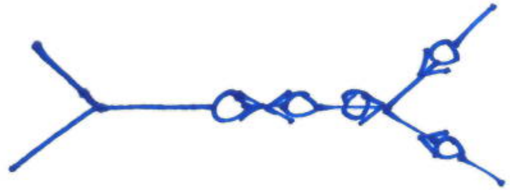
$$B_{N'} \xleftrightarrow{\text{BRAUER-SCHUR-WEYL}} SO_{2n}$$

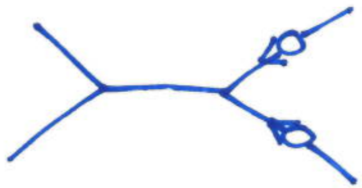
$\underline{g_2}$: SUBGROUP OF SO(7) PRESERVING (A/S) F_{ijk} , 

:

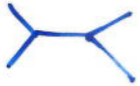
$M \in G_2 \Rightarrow$  = 

SO  IS IN CENTRALIZER :

 =  = 

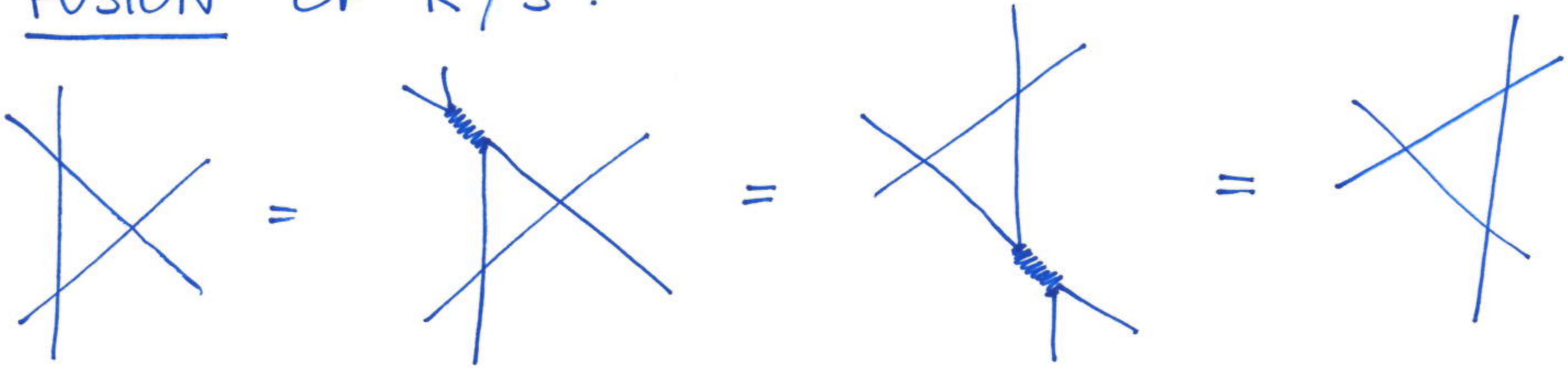
SINCE $M \in SO(7)$ = 

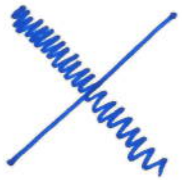
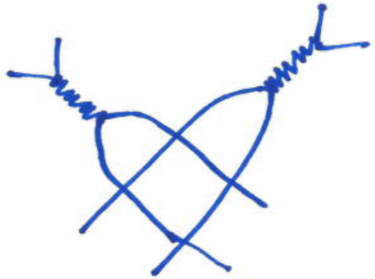
$$R(u) = P_{27} + \frac{u+1}{u-1}\left(P_{14} + \frac{u+6}{u-6} P_1\right) + \frac{u+4}{u-4} P_7$$

$P_{27} = \frac{1}{2}\left(= + \times\right) - \frac{1}{7}\,)($ $\quad P_{14} = \frac{1}{2}\left(= - \times\right) -$  $\quad P_1 = \frac{1}{7}\,)($ $\quad P_7 =$ 

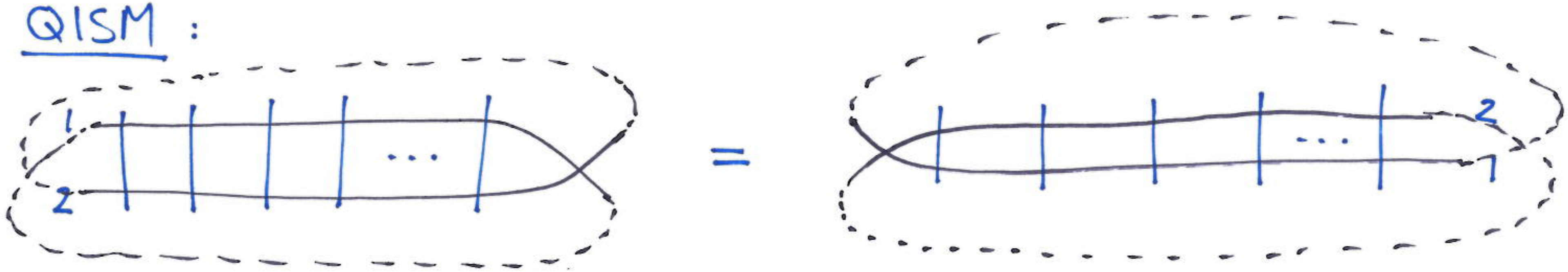
FUSION OF R / S :

KULISH SKLYANIN RESHETIKHIN '79



CONSTRUCTS  ; SIMILARLY  ETC .

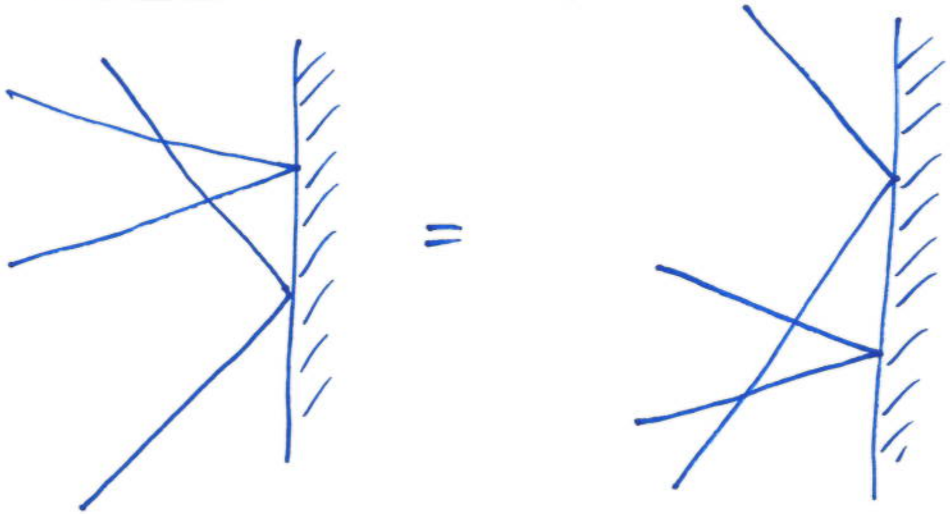
QISM :



AND TAKE TRACE TO OBTAIN COMMUTING CHARGES

....

BOUNDARY YBE

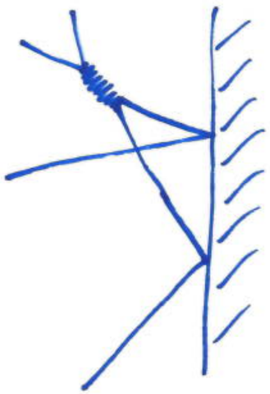


BULK $\mathfrak{g}$ ⇣ YANGIAN $Y(\mathfrak{g})$

BOUNDARY $\mathfrak{h} < \mathfrak{g}$ ⇣ TWISTED YANGIAN $Y(\mathfrak{g}, \mathfrak{h})$

$(\mathfrak{g}, \mathfrak{h})$ A SYMMETRIC PAIR

FUSION



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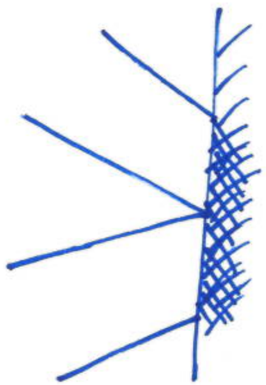
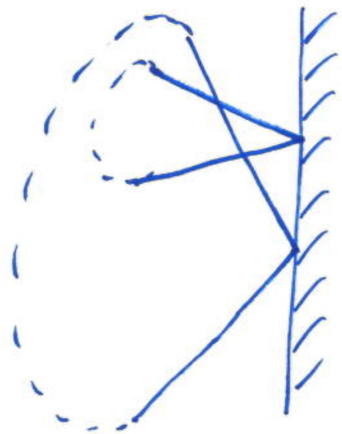
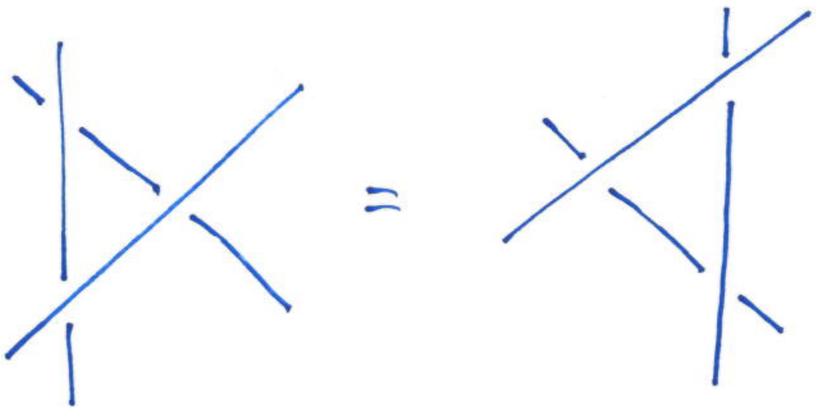


CHARGES



YBE AS BRAIDING



$$PR = \left(qx - \tfrac{1}{q}\right) P_+ - \left(\tfrac{x}{q} - q\right) P_-$$

$$P_+ = \frac{1}{1+q^2}\left(\asymp + q \times \right)$$

$$P_- = \frac{1}{1+q^2}\left(q^2 \asymp - q \times \right)$$

<u>NOTE</u> :

$$q P_+ - \frac{1}{q} P_- = \times$$

$$P_\pm^2 = P_\pm$$

FOLLOWS FROM SKEIN RELATION

$$\times\times - \left(q - \tfrac{1}{q}\right) \times - \asymp = 0$$

OR

$$\left(\times - q \asymp\right)\left(\times + \frac{1}{q} \asymp\right) = 0$$

TRACE

$\Rightarrow$ JONES POLYNOMIAL

SCHUR-WEYL DUAL	BULK		BOUNDARY
RATIONAL	YANGIANS	$\mathbb{C}S_N$ B_N (EXCEPTIONAL)	TWISTED YANGIANS DAHAs CHEREDNIK
TRIG.	QUANTIZED AFFINE ALGEBRAS "QUANTUM GROUPS"	IWAHORI HECKE BWM KUPERBERG SPIDER	VARIOUS! REGELSKIS & VLAAR
ELLIPTIC	SKLYANIN ALGEBRAS		

NJM, "INTRODUCTION TO YANGIAN...." ARXIV: hep-th/0409183

GWD+NJM, "AFFINE QUANTUM GROUPS" math.QA/0607228