Solving the NC $\Phi^3_{2,4,6}$ *models*

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History

- 80´ NCG - NC spaces — Alain Connes
- 92 Fuzzy Sphere regularize 2d QFT: — HG-Madore,
- 94 quantum st, Minkowski fields — Doplicher-Fredenhagen-Roberts
- 99 One-loop renormalization — Krajewski-Wulkenhaar;
  IR/UV Mixing — Minwalla-van Ramsdonk-Seiberg
  Connection to Strings: — Schomerus; Seiberg-Witten;...
- 00... — HG + R Wulkenhaar
- Model on Moyal space $(a \star b)(x) = \int d^D y d^D k a(x + \frac{1}{2}\Theta \cdot k) b(x+y)\, e^{iky}$

  $$S = \int d^4x \Big(\frac{Z}{2} \partial_\mu \phi \star \partial^\mu \phi + \frac{Z\mu^2}{2} \phi \star \phi + \frac{\lambda Z^2}{4} \phi \star \phi \star \phi \star \phi \Big)(x)$$

  is nonrenormalizable (mixing)
- 02 detailed... — Bahns-Doplicher-Fredenhagen-Piacitelli

NC toy Model: HG+R Wulkenhaar

- $S = \int d^4x \Big(\frac{Z}{2} \phi \star (\Delta + 4\Omega^2 (\Theta^{-1} x)^2 + \mu^2) \phi + \frac{\lambda Z^2}{4} \phi \star \phi \star \phi \star \phi \Big)(x)$
- four bare constants: Z, μ, λ, Ω
- there is a duality: Ω goes to $1/\Omega$ $\Omega = 1$ fixed point,
- RG flow bounded (pert. th) Disertori-Gurau-Magnen-Rivasseau
- At $\Omega = 1$, model becomes a MATRIX MODEL

$$S[\Phi] = Tr(ZE\Phi^2 + \frac{\lambda Z^2}{4} \Phi^4)$$

- Are dynamical matrix models, infinite nr of WI´s from action of $U(\infty)$.
- Schwinger-Dyson equations decouple

Matrix models: Relations

- 2 dim gravity - Hermitian Matrix Models $S[\phi] = \sum_n \frac{t_n}{n+1} Tr\,\phi^{n+1}$
- Intersection th on the moduli space of Riemann surfaces -top gravity
- Kontsevich Model: asymptotic expansions

  Miwa: $t_n = -(2n-1)!!TrE^{-(2n+1)}$ $S[\Phi] = Tr(ZE\Phi^2 + \lambda\frac{Z^{\frac{3}{2}}}{3}\Phi^3)$
- we add: NC Φ^3 model: needs Renormalization 05 HG+H Steinacker (and Φ^4)

$$S = \sum_{m,n\in\mathbb{N}^3_\Lambda} \frac{Z}{2}\phi_{mn}(E_n + E_m)\phi_{nm} + V(\phi)\,, \qquad E_n = \left(\frac{\mu^2_{bare}}{2} + |n|\right)\,,$$

$$|n| = n_1 + n_2 + n_3$$

$$V(\phi) = Z^{\frac{1}{2}} \sum_{n\in\mathbb{N}^3_\Lambda} (\alpha + \beta E_n + \gamma E_n^2)\Phi_{n,n} + \frac{Z^{3/2}\lambda}{3} \sum_{m,n,k,l\in\mathbb{N}^3_\Lambda} \phi_{mn}\phi_{nk}\phi_{kl}\,,$$

Two independent dimensions

1. Topological dimension 2 from expansion of matrix models into ribbon graphs, i.e. simplicial 2-complexes.
   - dual to triangulations (Φ^3) or quadrangulations (Φ^4) of 2D-surfaces
   - partition function counts them = 2D quantum gravity
   - non-planar ribbon graphs suppressed in large-$\mathcal{N}$ limit
2. Dynamical dimension D encoded in spectrum of the unbounded positive operator E,

$$D = \inf\{p \in \mathbb{R}_+ \; : \; \mathrm{Tr}((1+E)^{-\frac{p}{2}}) < \infty\}$$

   - ignored in 2D quantum gravity
   - highly relevant for renormalisation of matricial QFT

polynomial	finite	super-ren	just ren.	not ren.
Φ^3	$D < 2$	$2[\frac{D}{2}] \in \{2,4\}$	$2[\frac{D}{2}] = 6$	$2[\frac{D}{2}] > 6$
Φ^4	$D < 2$	$2[\frac{D}{2}] = 2$	$2[\frac{D}{2}] = 4$	$2[\frac{D}{2}] > 4$

Ward-Takahashi identity

- inner automorphism $\phi \mapsto U\phi U^\dagger$ of M_Λ, infinitesimally $\phi_{mn} \mapsto \phi_{mn} + \mathrm{i}\sum_{k\in\mathbb{N}^2_\Lambda}(B_{mk}\phi_{kn} - \phi_{mk}B_{kn})$
- not a symmetry of the action, but invariance of measure $\mathcal{D}\phi = \prod_{m,n\in\mathbb{N}^2_\Lambda} d\phi_{mn}$ gives

$$0 = \frac{\delta W}{\mathrm{i}\delta B_{ab}} = \frac{1}{\mathcal{Z}}\int \mathcal{D}\phi \,\Big(-\frac{\delta S}{\mathrm{i}\delta B_{ab}} + \frac{\delta}{\mathrm{i}\delta B_{ab}}(\mathrm{tr}(\phi J))\Big)e^{-S+\mathrm{tr}(\phi J)}$$
$$= \frac{1}{\mathcal{Z}}\int \mathcal{D}\phi \sum_n \Big((E_b - E_a)\phi_{bn}\phi_{na} + (\phi_{bn}J_{na} - J_{bn}\phi_{na})\Big)e^{-S+\mathrm{tr}(\phi J)}$$

where $W[J] = \ln \mathcal{Z}[J]$ generates connected functions

trick $\phi_{mn} \mapsto \frac{\partial}{\partial J_{nm}}$

$$0 = \Big\{\sum_n \Big((E_b - E_a)\frac{\delta^2}{\delta J_{nb}\,\delta J_{an}} + \Big(J_{na}\frac{\delta}{\delta J_{nb}} - J_{bn}\frac{\delta}{\delta J_{an}}\Big)\Big) \times \exp\Big(-V\big(\tfrac{\delta}{\delta J}\big)\Big)e^{\frac{1}{2}\sum_{p,q}J_{pq}(E_p+E_q)^{-1}J_{qp}}\Big\}_c$$

Definition of correlation functions...$B \geq 1, g \geq 0$

First terms of the expansion of the partition function:

$$\frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} = 1 + V \sum_{m} G_{|m|} J_{mm} + \frac{V}{2} \sum_{m,n} G_{|mn|} J_{mn} J_{nm}$$
$$+ \sum_{m,n} \Big(\frac{1}{2} G_{|m|n|} + \frac{V^2}{2} G_{|m|} G_{|n|} \Big) J_{mm} J_{nn}$$
$$+ \frac{V}{3} \sum_{m,n,k} G_{|mnk|} J_{mn} J_{nk} J_{km} + \sum_{m,n,k} \Big(\frac{1}{2} G_{|mn|k|} + \frac{V^2}{2} G_{|mn|} G_{|k|} \Big) J_{mn} J_{nm} J_{kk}$$
$$+ \sum_{m,n,k} \Big(\frac{1}{6V} G_{|m|n|k|} + \frac{V}{2} G_{|m|n|} G_{|k|} + \frac{V^3}{6} G_{|m|} G_{|n|} G_{|k|} \Big) J_{mm} J_{nn} J_{kk} + \dots .$$

All sums run from 0 to a cut-off $\mathcal{N}$.

Defines Greens functions $G_{|m_1|m_2|\dots|m_B|}$...$B \geq 1$

higher genus $G_{\dots} = \sum_{g=0}^{\infty} V^{-2g} G_{g,\dots}$

S-D, WT equs Φ^3_2, $D = 2$

action $S[\Phi] = V\,\mathrm{Tr}(E\Phi^2 + \alpha\Phi + \frac{\lambda}{3}\Phi^3)$ + R Wulkenhaar + A Sako

Strategy

use $\mathcal{Z}(J) = K \exp\left(-\frac{\lambda}{3V^2}\sum_{k,l,m=0}^{\mathcal{N}-1} \frac{\partial^3}{\partial J_{kl}\partial J_{lm}\partial J_{mk}}\right)\mathcal{Z}_{free}(J)$ (*)

to derive equations between Green functions

Inserting (*) into $G_{|a|} \equiv \frac{1}{V}\frac{\partial \log Z(J)}{\partial J_{aa}}\Big|_{J=0}$ gives

$$G_{|a|} = \frac{1}{2E_a}\left(\alpha - \lambda G^2_{|a|} - \frac{\lambda}{V}\sum_{m=0}^{\mathcal{N}-1} G_{|am|} - \frac{\lambda}{V^2}G_{|a|a|}\right)$$

- the equation is non-linear
- $\sum_{m=0}^{\infty} G_{|am|}$ diverges. Choose α such that $G_{|0|} = 0$:

Ward-Takahashi identity $G_{|ab|} = \frac{1}{E_a + E_b}\left(1 + \lambda\frac{G_{|a|} - G_{|b|}}{E_a - E_b}\right)$

Inserting $G_{|ab|}$ gives non-linear equation for $G_{|a|}$ alone, up to $\frac{1}{V^2}(G_{|a|a|} - G_{|0|0|})$ corrections which vanish for $V \to \infty$

Ward-Takahashi identity

Introduce $W_{|a|} := 2(\lambda G_{|a|} + E_a)$. Then:

1. $W_{|a|}^2 = 4E_a^2 - \frac{4\lambda^2}{V^2}(G_{|a|a|} - G_{|0|0|}) - \frac{2\lambda^2}{V}\sum_{m=0}^{\mathcal{N}-1}\left(\frac{W_{|a|} - W_{|m|}}{E_a^2 - E_m^2} - \frac{W_{|0|} - W_{|m|}}{E_0^2 - E_m^2}\right)$
2. $G_{|a_1 \dots a_N|} = \lambda \frac{G_{|a_1 a_3 \dots a_N|} - G_{|a_2 a_3 \dots a_N|}}{(E_{a_1}^2 - E_{a_2}^2)}$

Proposition: Solution of (2)

$$G_{|a_1 a_2, \dots a_N|} = \frac{\lambda^{N-2}}{2}\sum_{k=1}^{N} W_{|a_k|} \prod_{l=1, l \neq k}^{N} \frac{1}{E_{a_k}^2 - E_{a_l}^2}$$

Remains to solve (1) for $W_{|a|}$; in an $1/V^2$ expansion is $(G_{|a|a|} - G_{|0|0|})/(V^2)$ suppressed. One gets a non-linear (integral) equation for $W_{|a|}$ alone.

The non-linear integral equ and its solution: $D = 2$

Introduce density: $\rho(t) = \frac{1}{\mathcal{N}} \sum_{n=1}^{\mathcal{N}} \delta(t - (E_n - \frac{\mu^2}{2}))$

change variables $X := (2e(x) + 1)^2$, we have

$$W^2(X) + \int_1^\infty dY \rho(Y) \frac{W(X) - W(Y)}{X - Y} = X + \int_1^\infty dY \rho(Y) \frac{W(1) - W(Y)}{1 - Y} \qquad (*)$$

Theorem

The non-linear integral equation (*) is solved by (Makeenko,Semenoff)

$$W(X) := \sqrt{X + c} + \frac{1}{2} \int_1^\infty dY \frac{\rho(Y)}{(\sqrt{X + c} + \sqrt{Y + c})\sqrt{Y + c}}$$

where $c(\lambda)$ is the inverse solution of

$$W(1) = 1 = \sqrt{1 + c} + \frac{1}{2} \int_1^\infty dY \frac{\rho(Y)}{(\sqrt{1 + c} + \sqrt{Y + c})\sqrt{Y + c}}$$

All correlation functions for $N, B = 1, g = 0$ are explicitly known!

Holds for an interval of λ, Schwinger functions analytic

$B \geq 2, g = 0$

Solution for the $(1|1)$-point function $B = 2, g = 0$

$$W(X)G(X|Y) + \frac{1}{2}\int_1^\infty dZ\,\rho(Z)\,\frac{G(X|Y) - G(Z|Y)}{X - Z} = -\lambda G(X, Y, Y).$$

The $(1+1)$-point function

$$G(X|Y) = \frac{4\lambda^2}{\sqrt{X+c}\cdot\sqrt{Y+c}\cdot(\sqrt{X+c}+\sqrt{Y+c})^2},$$

The $(1|\ldots|1)$-point fct with $B \geq 3, g = 0$ is given by:

$$G(X^1|\ldots|X^B) = (-2\lambda)^{3B-4}\frac{d^{B-3}}{dt^{B-3}}\left(\frac{\frac{1}{\sqrt{X^1+c-2t}^3}\cdots\frac{1}{\sqrt{X^B+c-2t}^3}}{\left(1-\int_1^\infty \frac{dT\,\rho(T)}{\sqrt{T+c}}\frac{1}{(\sqrt{T+c}+\sqrt{T+c-2t})\sqrt{T+c-2t}}\right)^{B-2}}\right)\Bigg|_{t=0}$$

Bell polynomials were used...

Topological Recursion: Eynard, Orantin, Chekhov,...

Change of variables: $z := \sqrt{X+c}$ + R Wulkenhaar + A Hock

$$\mathcal{G}_g(z(X)) := W_g(X) \qquad \text{for the 1-point fct,}$$

$$\mathcal{G}_g(z_1^1(X_1^1),...,z_{N_1}^1(X_{N_1}^1)|...|z_1^B(X_1^B),...,z_{N_B}^B(X_{N_B}^B)) := G_g(X_1^1,...,X_{N_1}^1|...|X_1^B,...,X_{N_B}^B).$$

spectral curve:

$$\mathcal{G}_0(z) = z + \frac{1}{2}\int_{\sqrt{1+c}}^{\sqrt{\Lambda^2+c}} dy\, \frac{\tilde{\varrho}(y)}{(z+y)y}, \qquad \tilde{\varrho}(y) := 2y\,\varrho(\sqrt{y^2-c}),$$

$$\mathcal{G}_0(z_1|z_2) = \frac{4\lambda^2}{z_1z_2(z_1+z_2)^2}, \qquad \mathcal{G}_0(z_1|z_2|z_3) = -\frac{32\lambda^5}{\rho_0 z_1^3 z_2^3 z_3^3}.$$

$\tilde{\varrho}(y)$ has support in $[\sqrt{1+c}, \sqrt{\Lambda^2+c}] \subset \mathbb{R}_+$ because of $c > -1$.
$\mathcal{G}_0(z)$ extends to a sectionally holomorphic function with branch cut along $[-\sqrt{1+\Lambda^2}, -\sqrt{1+c}]$.

Topological Recursion

The $(1|1)$-point function of genus zero is holomorphic outside $z_i = 0$ and the diagonals $z_1 = -z_2$

$(1|1|1)$-point function (and all higher-B functions) at genus 0 are meromorphic with only pole at $z_i = 0$.

Let $\hat{K}_z$ be the integral operator of the linear integral equation,

$$\hat{K}_z f(z) := \mathcal{G}_0(z) f(z) + \frac{1}{2}\int_{\sqrt{1+c}}^{\sqrt{\Lambda^2+c}} dy\ \tilde{\varrho}(y) \frac{f(z)-f(y)}{z^2-y^2},$$

where $\mathcal{G}_0(z)$ is given.

In this notation, the top. rec. takes the form SD ... Tutte

$$\hat{K}_z \mathcal{G}_g(z) = -\frac{1}{2}\sum_{h=1}^{g-1} \mathcal{G}_h(z)\mathcal{G}_{g-h}(z) - 2\lambda^2 \mathcal{G}_{g-1}(z|z).$$

Boundary Creation Operator

For $J = \{1, \ldots, p\}$ let $|J| := p$ and $z_J := (z_1, \ldots, z_p)$. Define

$$\hat{\mathrm{A}}^{\dagger g}_{z_J,z} := \sum_{l=0}^{3g-3+p} \Big(-\frac{(3+2l)\varrho_{l+1}}{\varrho_0 z^3} + \frac{3+2l}{z^{5+2l}} \Big) \frac{\partial}{\partial \varrho_l} + \sum_{i \in J} \frac{1}{\varrho_0 z^3 z_i} \frac{\partial}{\partial z_i}.$$

Then

$$\mathcal{G}_g(z_1|...|z_B) = (2\lambda)^{3B-4} \hat{\mathrm{A}}^{\dagger g}_{z_1,\ldots,z_B} \big(\hat{\mathrm{A}}^{\dagger g}_{z_1,\ldots,z_{B-1}} \big(\cdots \hat{\mathrm{A}}^{\dagger g}_{z_1,z_2} \mathcal{G}_g(z_1)...\big)\big), \qquad z_i \neq 0,$$

where $\mathcal{G}_g(z_1)$ is the 1-point function of genus $g \geq 1$ and

$$\varrho_l := \delta_{l,0} - \frac{1}{2} \int_1^\infty \frac{dT\ \rho(T)}{(\sqrt{T+c})^{3+2l}}.$$

are moments of the (ren) measure.

Boundary Creation Annihilation operators: Bosonic

We introduce the operators

$$\hat{A}^\dagger_z := \sum_{l=0}^{\infty}\Big(-\frac{(3+2l)\varrho_{l+1}}{\varrho_0 z^3} + \frac{3+2l}{z^{5+2l}}\Big)\frac{\partial}{\partial \varrho_l}, \qquad \hat{N} = -\sum_{l=0}^{\infty}\varrho_l \frac{\partial}{\partial \varrho_l},$$

$$\hat{A}_{\check{z}} f(\bullet) := -\sum_{l=0}^{\infty} Res_{z\to 0}\Big[\frac{z^{4+2l}\varrho_l}{3+2l} f(z)dz\Big].$$

We call $\hat{A}_{\check{z}}$ a boundary annihilation operator acting on Laurent polynomials f. There is a unique function F_g of $\{\varrho_l\}$ satisfying $\mathcal{G}_g(z) = (2\lambda)^4 \hat{A}^\dagger_z F_g$,

$$F_1 = -\frac{1}{24}\log \varrho_0, \qquad F_g = \frac{1}{(2g-2)(2\lambda)^4}\hat{A}_{\check{z}}\mathcal{G}_g(\bullet) \text{ for } g \geq 1.$$

The F_g have for $g > 1$ a presentation as

$$F_g = (2\lambda)^{4g-4}\frac{P_{3g-3}(\varrho)}{\varrho_0^{2g-2}},$$

where $P_{3g-3}(\varrho)$ is a $(3g-3)$-weighted polynomial in $\{\frac{\varrho_1}{\varrho_0}, \dots, \frac{\varrho_{3g-3}}{\varrho_0}\}$.

Intersection Nrs

Let $t_{k+1} = -(2k+1)!!\rho_k$

$$F_g(t_0, t_2, t_3, \dots, t_{3g-2}) := \sum_{(k)} \frac{\langle \tau_2^{k_2} \tau_3^{k_2} \dots \tau_{3g-2}^{k_{3g-2}} \rangle}{(1-t_0)^{2g-2+\sum_i k_i}} \prod_{i=2}^{3g-2} \frac{t_i^{k_i}}{k_i!}, \quad \sum_{i \geq 2} (i-1)k_i = 3g-3,$$

be the generating function of intersection numbers of ceertain classes on the moduli spaces $\overline{\mathcal{M}}_{g,n}$ of complex curves of genus g. For any $N > 0$, the stable partition function satisfies

$$\exp\Big(\sum_{g=2}^{\infty} N^{2-2g} F_g(t)\Big) = \exp\Big(-\frac{1}{N^2}\Delta_t + \frac{F_2(t)}{N^2}\Big) 1$$

where $F_2(t) = \frac{7}{240} \cdot \frac{t_2^3}{3! T_0^5} + \frac{29}{5760} \frac{t_2 t_3}{T_0^4} + \frac{1}{1152} \frac{t_4}{T_0^3}$ with $T_0 := (1 - t_0)$
generates the intersection numbers of genus 2
Witten, Kontsevich, Itzykson-Zuber,...Mirzakhani...

Laplacian

$\Delta_t = -\sum_{i,j} \hat{g}^{ij}\partial_i\partial_j - \sum_i \hat{\Gamma}^i\partial_i$ is a Laplacian on the formal parameters $t_0, t_2, t_3, \ldots$ given by

$$\Delta_t := -\Big(\frac{2t_2^3}{45T_0^3} + \frac{37t_2t_3}{1050T_0^2} + \frac{t_4}{210T_0}\Big)\frac{\partial^2}{\partial t_0^2} - \Big(\frac{2t_2^3}{27T_0^4} + \frac{1097t_2t_3}{12600T_0^3} + \frac{41t_4}{2520T_0^2}\Big)\frac{\partial}{\partial t_0}$$
$$-\sum_{k=2}^{\infty}\Big(\Big(\frac{2t_2^2}{45T_0^3} + \frac{2t_3}{105T_0^2}\Big)t_{k+1} + \frac{t_2\mathcal{R}_{k+1}(t)}{2T_0} + \frac{3\mathcal{R}_{k+2}(t)}{2(3+2k)}\Big)\frac{\partial^2}{\partial t_k\partial t_0}$$
$$-\sum_{k,l=2}^{\infty}\Big(\frac{t_2t_{k+1}t_{l+1}}{90T_0^2} + \frac{t_{k+1}\mathcal{R}_{l+1}(t)}{4T_0} + \frac{t_{l+1}\mathcal{R}_{k+1}(t)}{4T_0} + \frac{(1{+}2k)!!(1{+}2l)!!\mathcal{R}_{k+l+1}(t)}{4(1{+}2k{+}2l)!!}\Big)\frac{\partial}{\partial t_k}$$
$$-\sum_{k=2}^{\infty}\Big(\Big(\frac{19t_2^2}{540T_0^4} + \frac{5t_3}{252T_0^3}\Big)t_{k+1} + \frac{t_2\mathcal{R}_{k+1}(t)}{48T_0^2} + \frac{\mathcal{R}_{k+2}(t)}{16(3+2k)T_0} + \frac{t_2t_{k+2}}{90T_0^3} + \frac{\mathcal{R}_{k+2}(t)}{2T_0}\Big)$$

$$\mathcal{R}_m(t) := \frac{2}{3}\sum_{k=1}^{m}\frac{(2m{-}1)!!\,kt_{k+1}}{(2k{+}3)!!T_0}\sum_{l=0}^{m-k}\frac{l!}{(m{-}k)!}B_{m-k,l}\Big(\Big\{\frac{j!t_{j+1}}{(2j{+}1)!!T_0}\Big\}_{j=1}^{m-l+1}\Big).$$

$F_g(t)$ are recursively extracted: $\mathcal{Z}_g(t) := \frac{1}{(g-1)!}(-\Delta_t + F_2(t))^{g-1}1$ Computer...

Summary

Renormalised Φ^3_D-QFT model on nc geometry of dimension $D \leq 6$ is solved

- Compute the free energy $F_g(t)$. It encodes $p(3g-3)$ intersection nrs.
- $F_1 = -\frac{1}{24} \log T_0$ for $g = 1$.
- Change variables to $\varrho_0 = 1 - t_0$, $\varrho_l = -\frac{t_{l+1}}{(2l+1)!!}$.
- ϱ_l are given by moments of the measure, c is implicitly defined.
- Apply to $F_g(\varrho)$ the boundary creation operators

Φ^3_D-model provides a fascinating toy model for a qft

- needs perturbative ren, needs Zimmermann's forest formula
- Resummation in genus? Is $\sum_{g=2}^{\infty} \mathcal{N}^{2-2g} F_g(t)$ Borel summable
- Integrability: KdV, deformed Virasoro-Witt algebra?